Asymptotic classes

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Outline of the talk

- Notation
- History and motivation
- Multidimensional asymptotic classes
- Ultraproducts and measurable structures
- Open problems

Notation

$\mathcal{L}$ will denote a (countable) finitary first-order language, $\mathcal{M}$ a finite $\mathcal{L}$-structure and $\mathcal{C}$ an (infinite) class of finite $\mathcal{L}$-structures. We will often distinguish between the $\mathcal{L}$-structure $\mathcal{M}$ and its underlying set M. We use $x, y, z, x_1, x_2, \ldots$ to denote variables and $a, b, c, a_1, a_2, \ldots$ to denote constant symbols. We will be lax and conflate constant symbols and their interpretations in a structure, i.e. we will write a for $a^{\mathcal{M}}$. We write $\bar{x}$ and $\bar{a}$ to denote finite tuples of variables and constant symbols respectively. We write $l(\bar{x})$ to denote the length of a tuple, e.g. if $\bar{x} = (x_1, \ldots, x_n)$, then $l(\bar{x}) = n$. We define $\mathbb{N}^+ := \{n \in \mathbb{N} : n \geq 1\}$, $\mathbb{R}^{\geq 0} := \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}^{>0}$, $\mathbb{Q}^{\geq 0}$, etc. similarly. For an $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ with $l(\bar{x}) = n$ and $l(\bar{y}) = m$, an $\mathcal{L}$-structure $\mathcal{M}$ and a tuple $\bar{a} \in M^m$, we define

$$\varphi(\mathcal{M}^n, \bar{a}) := \{\bar{b} \in M^n : \mathcal{M} \models \varphi(\bar{b}, \bar{a})\}.$$

History and motivation

The study of asymptotic classes stems from a deep application by Chatzidakis, van den Dries and Macintyre (CDM) in [2] of the Lang–Weil estimates [6] and the work of Ax [1]:

Theorem (CDM, 1992)

Let $\varphi(\bar{x}, \bar{y})$ be a formula in the language of rings $\mathcal{L}_{\text{ring}} = \{0, 1, +, \cdot\}$, where $l(\bar{x}) = n$ and $l(\bar{y}) = m$. Then there exist a constant $C \in \mathbb{R}^{>0}$ and a finite set D of pairs $(d, \mu) \in \{0, \ldots, n\} \times \mathbb{Q}^{>0}$ such that for every finite field $\mathbb{F}_q$ and for every $\bar{a} \in {\mathbb{F}_q}^m$, if $\varphi({\mathbb{F}_q}^n, \bar{a}) \neq \varnothing$, then

$$\left| |\varphi({\mathbb{F}_q}^n, \bar{a})| - \mu q^d \right| \leq C q^{d - 1/2} \qquad (*)$$

for some pair $(d, \mu) \in D$. Furthermore, the parameters are definable; that is, for each $(d, \mu) \in D$ there exists an $\mathcal{L}_{\text{ring}}$-formula $\varphi_{(d,\mu)}(\bar{y})$ such that for every $\mathbb{F}_q$, $\mathbb{F}_q \models \varphi_{(d,\mu)}(\bar{a})$ iff $\bar{a}$ satisfies $(*)$ for (d, μ).

N-dimensional asymptotic classes

Dugald Macpherson and Charles Steinhorn investigated other classes of finite structures that satisfy the CDM theorem. [7] To this end they defined the notion of an *asymptotic class* as a generalisation of the CDM theorem. The definition given below is that given by Richard Elwes in [4], which is itself a slight generalisation of the original definition in [7].

N-dimensional asymptotic classes

Definition (Macpherson–Steinhorn, Elwes)

Let $\mathcal{L}$ be a first-order language, $N \in \mathbb{N}^+$ and $\mathcal{C}$ a class of finite $\mathcal{L}$-structures. Then $\mathcal{C}$ is an *N-dimensional asymptotic class* iff for every $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, where $l(\bar{x}) = n$ and $l(\bar{y}) = m$,

(a) there exist a finite set $D \subset (\{0, \dots, Nn\} \times \mathbb{R}^{>0}) \cup \{(0,0)\}$ and a partition $\{\Phi_{(d,\mu)} : (d,\mu) \in D\}$ of the set $\Phi := \{(\mathcal{M}, \bar{a}) : \mathcal{M} \in \mathcal{C}, \bar{a} \in M^m\}$ such that for each $(d,\mu) \in D$

$$\left| |\varphi(\mathcal{M}^n, \bar{a})| - \mu |M|^{d/N} \right| = o\left(|M|^{d/N} \right)$$

for all $(\mathcal{M}, \bar{a}) \in \Phi_{(d,\mu)}$ as $|M| \to \infty$; and

(b) for each $(d,\mu) \in D$ there exists an $\mathcal{L}$-formula $\varphi_{(d,\mu)}(\bar{y})$ such that for every $\mathcal{M} \in \mathcal{C}$, $\mathcal{M} \models \varphi_{(d,\mu)}(\bar{a})$ iff $(\mathcal{M}, \bar{a}) \in \Phi_{(d,\mu)}$.

N-dimensional asymptotic classes

We call (a) the *size clause* and (b) the *definability clause*. If a class $\mathcal{C}$ satisfies (a) but not necessarily (b), then we call it a *weak N-dimensional asymptotic class*. We refer to the functions $\mu|\cdot|^{d/N}$ as *dimension–measure functions*.

The precise meaning of the *o*-notation is as follows: for every $\varepsilon > 0$ there exists $Q \in \mathbb{N}$ such that for all $(\mathcal{M}, \bar{a}) \in \Phi_{(d,\mu)}$, if $|M| > Q$, then

$$\left| |\varphi(\mathcal{M}^n, \bar{a})| - \mu|M|^{d/N} \right| \leq \varepsilon|M|^{d/N}$$

or, equivalently (since $|M|^{d/N} \neq 0$),

$$\frac{\left| |\varphi(\mathcal{M}^n, \bar{a})| - \mu|M|^{d/N} \right|}{|M|^{d/N}} \leq \varepsilon.$$

Some examples of N-dimensional asymptotic classes

- The class of finite fields ($N = 1$). [2]
- The class of finite cyclic groups ($N = 1$). [7, Theorem 3.14]
- Various other group- and graph-theoretic examples ($N = 1$). [7, Examples 3.3–3.6, Proposition 3.11]
- Families of finite difference fields $\{(\mathbb{F}_{p^{nk+m}}, \sigma^k) : k \in \mathbb{N}\}$, where p is prime, $m, n \in \mathbb{N}$ and σ is the Frobenius automorphism ($N = 1$). [Ryten, PhD thesis; see [4, §4]]
- For any smoothly approximable structure $\mathcal{M}$ there exists a subset of the set of finite envelopes of $\mathcal{M}$ that forms a rk($\mathcal{M}$)-dimensional asymptotic class. [4, Proposition 4.1] (The notion of smooth approximation goes back to Lachlan and was developed in great depth by Cherlin and Hrushovski in [3]. We omit the definition; see [5, §4] for a concise description.)
- Any family of non-abelian finite simple groups of a fixed Lie rank, where N varies depending on the family. [Ryten, PhD thesis; see [5, Theorem 6.1]]

See [4], [5], [7] and [8] for further examples, results and exposition.

Multidimensional asymptotic classes

An N-dimensional asymptotic class consists of structures of a fixed dimension N. However, there is no a priori reason why the CDM phenomenon shouldn't occur in classes where the dimensions of the structures vary or where the structures themselves consist of different orthogonal/independent parts of different dimensions. With this thought in mind, Macpherson and Steinhorn have developed a further generalisation of an asymptotic class, a so-called *multidimensional asymptotic class* – or *mac* for short. The precise details of the definition of a mac have yet to be finalised, and Will Anscombe and I have come up with a variation of the Macpherson–Steinhorn definition that we feel captures CDM phenomena in broad generality (but isn't so general as to be trivial). We first need to cover some preliminaries before we state the definition.

Multidimensional asymptotic classes

Consider a class $\mathcal{C}$ of finite $\mathcal{L}$-structures and let

$$\Phi := \{(\mathcal{M}, \bar{a}) : \mathcal{M} \in \mathcal{C}, \bar{a} \in M^m\}.$$

Definition

Let $\{\Phi_i : i \in I\}$ be a partition of Φ. The set Φ_i is said to be *definable* iff there exists an $\mathcal{L}$-formula $\psi(\bar{y})$ with $l(\bar{y}) = m$ such that for every $\mathcal{M} \in \mathcal{C}$ and every $\bar{a} \in M^m$, $(\mathcal{M}, \bar{a}) \in \Phi_i$ iff $\mathcal{M} \models \psi(\bar{a})$. The partition is said to be *definable* iff Φ_i is definable for every $i \in I$ and to be *finite* iff the indexing set I is finite.

Multidimensional asymptotic classes

Definition

Let R be any set of functions from $\mathcal{C}$ to $\mathbb{R}^{\geq 0}$. The class $\mathcal{C}$ is a *multidimensional asymptotic class for R in $\mathcal{L}$* iff for any $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, where $l(\bar{x}) = n$ and $l(\bar{y}) = m$, there exist a finite definable partition $\Phi_1, \ldots, \Phi_k$ of Φ and functions $h_1, \ldots, h_k \in R$ such that for each $i \in \{1, \ldots, k\}$

$$\Big| |\varphi(\mathcal{M}^n, \bar{a})| - h_i(\mathcal{M}) \Big| = o(h_i(\mathcal{M})) \qquad (1)$$

for all $(\mathcal{M}, \bar{a}) \in \Phi_i$ as $|M| \to \infty$.

For brevity we talk of an *R-mac in $\mathcal{L}$*.

Multidimensional asymptotic classes

The idea behind this definition is go to the heart of the CDM phenomenon, namely the phenomenon of **finitely** many functions describing the sizes of **in**finitely many definable sets (in a definable way). We have thus abandoned the dimension–measure functions, as they are secondary to this more fundamental property. Although the definition is very general and allows R to be *any* set of functions from $\mathcal{C}$ to $\mathbb{R}^{\geq 0}$, the motivating examples take R to be a set of functions defined in terms of the sizes of certain parts of the structures, such as sorts or equivalence classes. However, there may be examples where the functions in R are much more exotic.

Examples of multidimensional asymptotic classes

- Every N-dimensional asymptotic class is an R-mac.
- In soon-to-be-submitted work, Darío García, Macpherson and Steinhorn have shown that the class of two-sorted structures consisting of a finite field F in the language of rings and a finite-dimensional vector space V over F in the language of additive groups with a map $F \times V \to V$, where the dimension and the field both vary freely, forms an R-mac, where the functions in R are rational polynomials in the sizes of the sorts. This class remains an R-mac if each structure is equipped with a bilinear form.
- In the same manuscript, García, Macpherson and Steinhorn have shown that for any fixed prime p, the class of groups $\{(\mathbb{Z}/p^n\mathbb{Z})^m : m, n > 0\}$ forms an R-mac. The functions in R are integer polynomials in p with exponents depending on m and n. This class is in fact an *exact multidimensional class*: the functions in R give the precise sizes of the definable sets, not just bounded approximations.

A non-example

The class of **rings** $\{(\mathbb{Z}/p^n\mathbb{Z})^m : m, n > 0\}$ is **not** an R-mac. This is because the formula $\varphi(x, y) :\equiv \exists z\,(z \cdot y = x)$ can pick out unboundedly many subsets of $\mathbb{Z}/p^n\mathbb{Z}$ of different sizes, since $|\varphi(\mathbb{Z}/p^n\mathbb{Z}, p^i)| = p^{n-i}$ for $i \in \{0, \ldots, n\}$, and thus no **finite** set of functions $h_1, \ldots, h_l \in R$ is able to approximate the sizes of these definable sets for all n.

Some structural results

- The Projection Lemma: Suppose that the definition of an R-mac holds for $\mathcal{C}$, R and all $\mathcal{L}$-formulae $\varphi(x, \bar{y})$ with a single variable x. Then $\mathcal{C}$ is an $\langle R \rangle$-mac in $\mathcal{L}$, where $\langle R \rangle$ is the semi-ring generated by R under addition and multiplication in $\mathbb{R}$.
- If a class $\mathcal{C}'$ is bi-interpretable with an R-mac $\mathcal{C}$, then $\mathcal{C}'$ is an $\langle R \rangle_{\mathrm{div}}$-mac, where $\langle R \rangle_{\mathrm{div}}$ is the semi-ring generated by R under addition, multiplication and division in $\mathbb{R}$. (This is an adaption of a result of Elwes in [4].)

Ultraproducts and measurable structures

So far we have talked only about classes of finite structures, but an important infinite counterpart to asymptotic classes are so-called *measurable structures*. As well as being interesting objects in their own right, measurable structures are invaluable for proving theorems about ultraproducts of asymptotic classes, as any infinite ultraproduct of an asymptotic class is a measurable structure. [5, Proposition 3.9]

Just as the older notion of an N-dimensional asymptotic class has been generalised to that of an R-mac, the concept of a measurable structure has been generalised to that of a T*-measurable structure*. We will cover only this more recent notion. Details of the original notion can be found in [5, §3] and [8, §5].

T-measurable structures

Definition

Let T be a semi-ring of characteristic zero and for an $\mathcal{L}$-structure $\mathcal{M}$ let Def($\mathcal{M}$) denote the set of all definable subsets of $\mathcal{M}^n$ for all $n > 0$. An infinite $\mathcal{L}$-structure $\mathcal{M}$ is *T-measurable* iff there exists a function $H\colon \mathrm{Def}(\mathcal{M}) \to T$ that satisfies the following conditions:

(a) For every $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, where $l(\bar{x}) = n$ and $l(\bar{y}) = m$, there exist $f_1, \ldots, f_k \in T$ such that for each $\bar{a} \in M^m$, $H(\varphi(\mathcal{M}^n, \bar{a})) = f_i$ for some $i \in \{1, \ldots, k\}$. Moreover, the set $\{\bar{a} \in M^m : H(\varphi(\mathcal{M}^n, \bar{a})) = f_i\}$ is $\varnothing$-definable for each f_i.

(b) $H(X) = |X|$ for all finite $X \in \mathrm{Def}(\mathcal{M})$ and $H(X_1 \cup \cdots \cup H_r) = H(X_1) + \cdots + H(X_r)$ for all disjoint $X_1, \ldots, X_r \in \mathrm{Def}(\mathcal{M})$.

(c) (Fubini) For all $X, Y \in \mathrm{Def}(\mathcal{M})$, if $\rho\colon X \to Y$ is a definable surjection and $H(\rho^{-1}(y)) = f$ for all $y \in Y$, then $H(X) = f \cdot H(Y)$.

T-measurable structures

As with N-dimensional asymptotic classes and measurable structures, any infinite ultraproduct of an R-mac is a T-measurable structure, where T is built using R. Skipping over the details, the idea is that for a given $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, the functions $h_1, \ldots, h_k \in R$ give rise to the functions $f_1, \ldots, f_k \in T$.

By moving to an infinite structure we can call upon the techniques and results of infinite model theory, bringing clear advantages.

Some results using T-measurable structures

- No ultraproduct of an R-mac has the strict order property.
- If the set of functions R is isomorphic to the semi-ring $\mathbb{R}^{\geq 0}[X_1, \ldots, X_n]$, then any infinite ultraproduct of an R-mac is supersimple.
- Recall the previous example of finite-dimensional vector spaces over finite fields. The ultraproduct of this class is supersimple. However, if we add a bilinear form, then the ultraproduct is no longer supersimple, although it is still NTP1.

Open problems

- We have found a sufficient condition on R for an infinite ultraproduct of an R-mac to be supersimple, but can we find a necessary condition?
- What conditions, if any, can we place on R to ensure that an infinite ultraproduct is NTP1?
- We conjecture that the set of *all* finite envelopes of *any* smoothly approximable structure forms an exact multidimensional class. Can we prove this?
- Can we find new, interesting examples of R-macs, especially ones that make full use of the generality of R?

Thank you for your attention!

Slides available at www.dwood.eu/research.

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