Exact classes and smooth approximation

Daniel Wolf

Joint work with Sylvy Anscombe (UCLan),
Dugald Macpherson (Leeds) and Charles Steinhorn (Vassar)

Midlands Logic Seminar

Warwick Mathematics Institute, Saint Patrick's Day 2016

Outline of the talk

- History and motivation
- *R*-macs and *R*-mecs
- Smooth approximation
- Macpherson's conjecture

The motivating example

The study of asymptotic classes stems from a deep application by Chatzidakis, van den Dries and Macintyre (CDM) in [3] of the Lang–Weil estimates [10] and the work of Ax [2]:

Theorem (CDM, 1992)

Let $\varphi(\bar{x}, \bar{y})$ be a formula in the language of rings $\mathcal{L}_{\text{ring}} = \{0, 1, +, \cdot\}$, where $l(\bar{x}) = n$ and $l(\bar{y}) = m$. Then there exist a constant $C \in \mathbb{R}^{>0}$ and a finite set D of pairs $(d, \mu) \in \{0, \dots, n\} \times \mathbb{Q}^{>0}$ such that for every finite field $\mathbb{F}_q$ and for every $\bar{a} \in {\mathbb{F}_q}^m$, if $\varphi({\mathbb{F}_q}^n, \bar{a}) \neq \varnothing$, then

$$\left| |\varphi({\mathbb{F}_q}^n, \bar{a})| - \mu q^d \right| \leq C q^{d - 1/2} \tag{$*$}$$

for some pair $(d, \mu) \in D$. Furthermore, the parameters are definable; that is, for each $(d, \mu) \in D$ there exists an $\mathcal{L}_{\text{ring}}$-formula $\varphi_{(d,\mu)}(\bar{y})$ such that for every $\mathbb{F}_q$, $\mathbb{F}_q \models \varphi_{(d,\mu)}(\bar{a})$ iff $\bar{a}$ satisfies $(*)$ for (d, μ).

N-dimensional asymptotic classes

Macpherson and Steinhorn investigated other classes of finite structures that satisfy the CDM theorem. [11] To this end they defined the notion of an *asymptotic class* as a generalisation of the CDM theorem. The definition given below is that given by Elwes in [6], which is itself a slight generalisation of the original definition in [11].

For a class $\mathcal{C}$ of $\mathcal{L}$-structures and an arbitrary $m \in \mathbb{N}^+$, define

$$\Phi := \{(\mathcal{M}, \bar{a}) : \mathcal{M} \in \mathcal{C}, \bar{a} \in M^m\}.$$

Borrowing a term from algebra, we sometimes refer to the elements of Φ as *pointed structures*.

N-dimensional asymptotic classes

Definition (Macpherson–Steinhorn, Elwes, 2007)

Let $\mathcal{L}$ be a first-order language, $N \in \mathbb{N}^+$ and $\mathcal{C}$ a class of finite $\mathcal{L}$-structures. Then $\mathcal{C}$ is an *N-dimensional asymptotic class* iff for every $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, where $l(\bar{x}) = n$ and $l(\bar{y}) = m$,

(a) there exist a finite set $D \subset (\{0, \ldots, Nn\} \times \mathbb{R}^{>0}) \cup \{(0, 0)\}$ and a partition $\{\Phi_{(d,\mu)} : (d, \mu) \in D\}$ of Φ such that for each $(d, \mu) \in D$,

$$\left| |\varphi(\mathcal{M}^n, \bar{a})| - \mu |M|^{d/N} \right| = o\left(|M|^{d/N} \right)$$

for all $(\mathcal{M}, \bar{a}) \in \Phi_{(d,\mu)}$ as $|M| \to \infty$; and

(b) for each $(d, \mu) \in D$ there exists an $\mathcal{L}$-formula $\varphi_{(d,\mu)}(\bar{y})$ such that for every $\mathcal{M} \in \mathcal{C}$, $\mathcal{M} \models \varphi_{(d,\mu)}(\bar{a})$ iff $(\mathcal{M}, \bar{a}) \in \Phi_{(d,\mu)}$.

N-dimensional asymptotic classes

We call (a) the *size clause* and (b) the *definability clause*. If a class $\mathcal{C}$ satisfies (a) but not necessarily (b), then we call it a *weak N-dimensional asymptotic class*. We refer to the functions $\mu|\cdot|^{d/N}$ as *dimension–measure functions*.

The precise meaning of the *o*-notation is as follows: for every $\varepsilon > 0$ there exists $Q \in \mathbb{N}$ such that for all $(\mathcal{M}, \bar{a}) \in \Phi_{(d,\mu)}$, if $|M| > Q$, then

$$\left| |\varphi(\mathcal{M}^n, \bar{a})| - \mu |M|^{d/N} \right| \leq \varepsilon |M|^{d/N}$$

or, equivalently (since $|M|^{d/N} \neq 0$),

$$\frac{\left| |\varphi(\mathcal{M}^n, \bar{a})| - \mu |M|^{d/N} \right|}{|M|^{d/N}} \leq \varepsilon.$$

Some examples of N-dimensional asymptotic classes

- The class of finite fields ($N = 1$). [3]
- The class of finite cyclic groups ($N = 1$). This is in fact an exact class (defined later). [11, Theorem 3.14]
- Some group- and graph-theoretic examples, in particular the class of Paley graphs ($N = 1$). [11, Examples 3.3–3.6, Proposition 3.11]
- Families of finite difference fields $\{(\mathbb{F}_{p^{nk+m}}, \sigma^k) : k \in \mathbb{N}\}$, where p is prime, $m, n \in \mathbb{N}$ and σ is the Frobenius automorphism ($N = 1$). [Ryten, PhD thesis; see [6, §4]]
- For any smoothly approximable structure $\mathcal{M}$ (defined later), there exists a subset of the set of finite envelopes of $\mathcal{M}$ that forms a rk($\mathcal{M}$)-dimensional asymptotic class. [6, Proposition 4.1]
- Any family of non-abelian finite simple groups of a fixed Lie rank, where N varies depending on the family. [Ryten, PhD thesis; see [7, Theorem 6.1]]

See [6], [7], [11] and [12] for further examples, results and exposition.

Multidimensional asymptotic classes

We have developed the notion of a multidimensional asymptotic class, a generalisation of an N-dimensional asymptotic class that captures more CDM-like behaviour. [1]

Multidimensional asymptotic classes

For a class $\mathcal{C}$ of finite $\mathcal{L}$-structures, recall the set Φ of pointed structures:

$$\Phi := \{(\mathcal{M}, \bar{a}) : \mathcal{M} \in \mathcal{C}, \bar{a} \in M^m\}.$$

Definition (Definable partition)

Let $\{\Phi_i : i \in I\}$ be a partition of Φ. The set Φ_i is said to be *definable* iff there exists an $\mathcal{L}$-formula $\psi_i(\bar{y})$ with $l(\bar{y}) = m$ such that for every $\mathcal{M} \in \mathcal{C}$ and every $\bar{a} \in M^m$, $(\mathcal{M}, \bar{a}) \in \Phi_i$ iff $\mathcal{M} \models \psi_i(\bar{a})$. The partition is said to be *definable* iff Φ_i is definable for every $i \in I$ and to be *finite* iff the indexing set I is finite.

Multidimensional asymptotic classes, aka *R*-macs

Definition (Anscombe, Macpherson, Steinhorn, W.)

Let R be any set of functions from $\mathcal{C}$ to $\mathbb{R}^{\geq 0}$. The class $\mathcal{C}$ is a *multidimensional asymptotic class for R in $\mathcal{L}$*, or *R-mac in $\mathcal{L}$* for short, iff for any $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, where $l(\bar{x}) = n$ and $l(\bar{y}) = m$, there exist a finite definable partition $\Phi_1, \ldots, \Phi_k$ of Φ and functions $h_1, \ldots, h_k \in R$ such that for each $i \in \{1, \ldots, k\}$,

$$\Big| |\varphi(\mathcal{M}^n, \bar{a})| - h_i(\mathcal{M}) \Big| = o(h_i(\mathcal{M})) \tag{1}$$

for all $(\mathcal{M}, \bar{a}) \in \Phi_i$ as $|M| \to \infty$.

The meaning of the o-notation is as before and we continue with the previous terminology of *size clause*, *definability clause* and *weak R-mac*.

Examples of *R*-macs

- Any *N*-dimensional asymptotic class.
- The class of all finite sets, where $\mathcal{L} = \varnothing$. This is in fact an exact class (defined later).
- The class $\{(\mathbb{Z}/p^n\mathbb{Z})^m : n, m \in \mathbb{N}^+\}$ of **groups**, where p is any prime and $\mathcal{L} = \{+\}$. [8] Note that this does not fit into the previous framework of *N*-dimensional asymptotic classes.

Non-examples of R-macs

- The class $\mathcal{C}$ of all finite linear orders in (any expansion of) the language $\mathcal{L} = \{<\}$ does not form an R-mac for any R.

  Proof. Let $\varphi(x, y)$ be the formula $x < y$ and consider the finite total order $\mathcal{M}_n = \{a_0 < a_1 < \cdots < a_n\}$. Then $|\varphi(\mathcal{M}_n, a_i)| = i$. So as we let n increase and let i vary we define arbitrarily many subsets of **distinct** sizes. Thus no **finite** number of functions from R can approximate $|\varphi(\mathcal{M}_n, a_i)|$ for all $n, i \in \mathbb{N}$. □

- Let p be prime. Then the class $\{(\mathbb{Z}/p^n\mathbb{Z})^m : n, m \in \mathbb{N}^+\}$ of **rings** in (any expansion of) the language $\mathcal{L} = \{+, \times\}$ does not form an R-mac for any R.

  Proof. Let $\varphi(x, y)$ be the formula $\exists z\,(x = z \times y)$. Then $|\varphi(\mathbb{Z}/p^n\mathbb{Z}, p^i)| = p^{n-i}$. So as we let n increase and let i vary we define arbitrarily many subsets of distinct sizes and thus no finite number of functions from R can approximate $|\varphi(\mathbb{Z}/p^n\mathbb{Z}, p^i)|$ for all $n, i \in \mathbb{N}$. (Notice that we didn't need to consider m.) □

R-mecs

There is a stronger notion of a *multidimensional exact class for R in* $\mathcal{L}$, or *R-mec in* $\mathcal{L}$ for short. This is where the previous definition holds, but where we have equality instead of the approximation (1) , i.e. for each $i \in \{1, \ldots, k\}$,

$$|\varphi(\mathcal{M}^n, \bar{a})| = h_i(\mathcal{M}) \tag{2}$$

for all $(\mathcal{M}, \bar{a}) \in \Phi_i$.

Note that we often refer to *R*-mecs as *exact classes*. Also note that while an *R*-mec is necessarily an *R*-mac, an *R*-mac need not be an *R*-mec. For example, the class of finite fields and the class of Paley graphs are both *R*-macs, but they are not *R*-mecs. [1]

Smooth approximation

Smooth approximation was invented by Lachlan the 1980s and then further developed by him and others, e.g. [4], [5] and [9].

Definition (Smooth approximation)

Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. $\mathcal{N}$ is a *homogeneous substructure* of $\mathcal{M}$, notationally $\mathcal{N} \subseteq_{\text{hom}} \mathcal{M}$, iff $\mathcal{N}$ is an $\mathcal{L}$-substructure of $\mathcal{M}$ and for every $k \in \mathbb{N}^+$ and every pair $\bar{a}, \bar{b} \in N^k$, $\bar{a}$ and $\bar{b}$ lie in the same $\text{Aut}(\mathcal{M})$-orbit iff $\bar{a}$ and $\bar{b}$ lie in the same $\text{Aut}_{\{N\}}(\mathcal{M})$-orbit, where $\text{Aut}_{\{N\}}(\mathcal{M}) := \{\sigma \in \text{Aut}(\mathcal{M}) : \sigma(N) = N\}$.

An $\mathcal{L}$-structure $\mathcal{M}$ is *smoothly approximable* iff $\mathcal{M}$ is $\aleph_0$-categorical and there exists a sequence $(\mathcal{M}_i)_{i<\omega}$ of finite $\mathcal{L}$-structures such that $\mathcal{M}_i \subseteq_{\text{hom}} \mathcal{M}$ and $M_i \subset M_{i+1}$ for all $i < \omega$ and $\bigcup_{i<\omega} M_i = M$. We say that $\mathcal{M}$ is *smoothly approximated* by the $\mathcal{M}_i$.

Examples of smoothly approximated structures

(1) Trivial example: Let $\mathcal{M}$ be a countably infinite set in the language of equality. Enumerate $\mathcal{M}$ by $(m_i : i < \omega)$ and let $\mathcal{M}_i = \{m_0, \ldots, m_i\}$. Then each $\mathcal{M}_i$ is a finite homogeneous substructure of $\mathcal{M}$ and $\mathcal{M} = \bigcup_{i<\omega} \mathcal{M}_i$.

(2) Let $\mathcal{M}$ be the unique countable structure consisting of infinitely many E_1-equivalence classes and a refinement E_2 such that each E_2-equivalence class is also infinite, i.e. first partition $\mathcal{M}$ into infinitely many E_1-classes and then partition each E_1-class into infinitely many infinite E_2-classes. Enumerate the E_1-classes by $(e_j : j < \omega)$ and the E_2-classes within each e_j by $(e_{jk} : k < \omega)$. Finally, enumerate the elements of each e_{jk} by $(e_{jkn} : n < \omega)$. Let $\mathcal{M}_i := \{e_{jkn} : j, k, n \leq i\}$. Then each $\mathcal{M}_i$ is a finite homogeneous substructure of $\mathcal{M}$ and $\mathcal{M} = \bigcup_{i<\omega} \mathcal{M}_i$.

(3) Let $\mathcal{M}$ be the direct sum of ω-many copies of $\mathbb{Z}/p^2\mathbb{Z}$, where p is some fixed prime. Let $\mathcal{M}_i$ consist of the first i copies of $\mathbb{Z}/p^2\mathbb{Z}$. Then each $\mathcal{M}_i$ is a finite homogeneous substructure of $\mathcal{M}$ and $\mathcal{M} = \bigcup_{i<\omega} \mathcal{M}_i$.

A link between smooth approximation and exact classes

Smoothly approximable structures provide a generic example of *R*-mecs:

Proposition (W.)

Let $\mathcal{M}$ be an $\mathcal{L}$-structure smoothly approximated by finite homogeneous substructures $(\mathcal{M}_i)_{i<\omega}$. Then there exists R such that $\{\mathcal{M}_i : i < \omega\}$ is an R-mec in $\mathcal{L}$.

The proof makes essential use of the Ryll-Nardzewski theorem and a result of Kantor–Liebeck–Macpherson in [9].

An obvious question is the following: What's *R*? This brings us to the work of Cherlin and Hrushovski in [5].

Cherlin–Hrushovski

Cherlin and Hrushsovski develop in [5] a very deep structure theory around $\aleph_0$-categoricity and smooth approximation. Key to this are the notions of Lie coordinatisation and quasifiniteness, which turn out to equivalent to smooth approximation. We state a theorem arising from [5] that is germane to our current work, namely an adapted version of Theorem 6 from that text:

Theorem (Cherlin–Hrushovski, 2003)

Let $\mathcal{L}$ be a finite language and $d \in \mathbb{N}^+$. Define $\mathcal{C}(\mathcal{L}, d)$ to be the class of all finite $\mathcal{L}$-structures with at most d 4-types. Then there is a finite partition $\mathcal{F}_1, \ldots, \mathcal{F}_k$ of $\mathcal{C}(\mathcal{L}, d)$ such that the structures in each $\mathcal{F}_i$ smoothly approximate an $\mathcal{L}$-structure $\mathcal{M}_i$. Moreover, the $\mathcal{F}_i$ are definably distinguishable: For each $\mathcal{F}_i$ there exists an $\mathcal{L}$-sentence χ_i such that for all $\mathcal{M} \in \mathcal{C}(\mathcal{L}, d)$ above some minimum size, $\mathcal{M} \models \chi_i$ if and only if $\mathcal{M} \in \mathcal{F}_i$.

Macpherson's conjecture

Another relevant result from [5] is Proposition 5.2.2, which provides precise information about the sizes of definable sets in finite homogeneous substructures. These two results from [5], together with the previous proposition, yield a proof of the following result, as conjectured by Macpherson, although some details still need to be worked out:

Theorem (almost)

Let $\mathcal{L}$ be a (finite?) language and let $d \in \mathbb{N}^+$. Define $\mathcal{C}(\mathcal{L}, d)$ to be the class of all finite $\mathcal{L}$-structures with at most d 4-types. Then $\mathcal{C}(\mathcal{L}, d)$ is an R-mec, where R is a semi-ring of polynomials in the sizes of the base finite fields.

Further properties of R can be given, but in order to state them we would need to go in a lot more detail.

Thank you for your attention!

Slides available at:

www.dwolf.eu/research

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