School of Mathematics
University of Leeds

Graduate Course

Around Stable Groups

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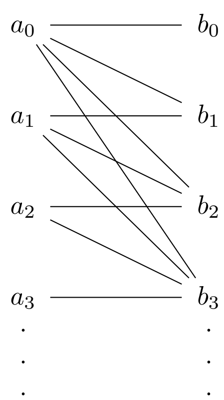

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Contents

Chapter 0

Introduction

These are the lecture notes of the graduate course 'Around Stable Groups' given by Anand Pillay between January and April 2013 in the School of Mathematics at the University of Leeds. The typists are PhD students who attended the course: Daniel Wood (mmdaw@leeds.ac.uk) typed up Chapter 1, Lovkush Agarwal (mmla@leeds.ac.uk) the majority of Chapter 2 and Ricardo Bello Aguirre (mmriba@leeds.ac.uk) the end of Chapter 2 and all of Chapter 3. The notes were revised slightly in October 2013; we plan to revise them further at some point in the future.

We will assume knowledge of elementary model theory, e.g. saturated models, homogeneity, types, indiscernibles, T^{eq}, $\mathcal{M}^{\mathrm{eq}}$, Tarski–Vaught, etc. We shall forgo set-theoretic concerns and assume the existence of saturated models of arbitrarily large cardinalities.

A lot of the material in these notes is based upon that in [5].[1] Another useful text is [1], from which the proof of Lemma 1.2 in these notes is adapted (see [1, Lemma 2.10]).

And finally, a note about the numbering: Numbered remarks from the lectures are numbered as they were in the lectures, while (some) other remarks are shown as 'Comments' and are labelled alphabetically.

[1] Note that there is some slight discrepancy between the notion of forking presented in [5] and that presented in some other texts, notably those of the French school.

Chapter 1

Local stability and stability

We start by setting out some general notation and terminology, which will apply throughout unless otherwise specified. $\mathcal{L}$ will be some first order-language, T will be a complete[1] $\mathcal{L}$-theory with infinite models and $\bar{\mathcal{M}}$ will be some sufficiently large saturated model of T, i.e. a monster model.[2] You may take T or $\mathcal{L}$ to be countable if you so wish. By a *model* we will (usually) mean a small elementary substructure $\mathcal{M} \prec \bar{\mathcal{M}}$ and, unless otherwise specified, $\mathcal{M}$ will denote a model $\mathcal{M} \prec \bar{\mathcal{M}}$. We write $\models \varphi(a)$ for $\bar{\mathcal{M}} \models \varphi(a)$, although we may sometimes pass to an even bigger model. We will usually use A to denote an arbitrary subset of $\mathcal{M}$ or $\bar{\mathcal{M}}$. $\mathcal{L}_A$ denotes the language $\mathcal{L}$ with additional constant symbols for the elements of A; we will often conflate $a \in A$ and its corresponding constant symbol in $\mathcal{L}_A$. We use the letters x, y, z for n-tuples of free variables and a, b, c for n-tuples of elements of $\bar{\mathcal{M}}$ (or $\mathcal{M}$ if specified), where $1 \leq n < \omega$. (Possibly) infinite tuples are denoted $\bar{a}, \bar{b}, \bar{c}$. $\varphi(x, y)$ will invariably denote an $\mathcal{L}$-formula (typically a stable one) whose free variables are among x and y. Other lower-case Greek letters will be used for other formulae and/or ordinals and cardinals. Types will usually be denoted by lower-case Roman letters $p, q, r, \ldots$ We write $a \in \mathcal{M}$ rather than $a \in \mathcal{M}^n$ or $a \in M^n$ if a is an n-tuple (we don't distinguish between the structure $\mathcal{M}$ and its domain M).

Local stability

Definition 1.1. A formula $\varphi(x, y)$ is *stable* (in T) iff there do not exist a_i, b_i for $i < \omega$ such that

$$\models \varphi(a_i, b_j) \text{ if and only if } i \leq j \tag{1.1}$$

for all $i, j < \omega$. A formula is *unstable* iff it is not stable.

We refer to this as *local* stability because it concerns just one formula $\varphi(x, y)$. *Stability* (without a modifier) concerns the local stability of all formulae in a theory; see footnote 27.

Comment 1.A.

(i) In Definition 1.1 we could write $i < j$ instead of $i \leq j$, the definitions would be equivalent.

(ii) The collection of stable formulae is closed under finite Boolean combinations and negations. (Henceforth 'Boolean combination' will mean 'finite Boolean combination'.)

φ and φ^*

(iii) $\varphi(x, y)$ is stable if and only if $\varphi^*(y, x)$ is stable, where $\varphi^*(y, x)$ is $\varphi(x, y)$.[3] So stability is symmetric in x and y.

[1] That is, $T \vdash \psi$ or $T \vdash \neg\psi$ for every sentence $\psi \in \mathcal{L}$.

[2] Note that some authors use $\mathbb{C}$, $\mathbf{C}$ or $\mathfrak{C}$ instead of $\bar{\mathcal{M}}$. We refrain from using this notation in order to avoid any confusion with the field of complex numbers.

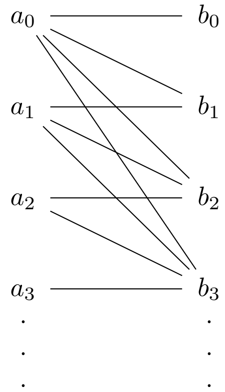
[3] This may seem completely trivial at first, since the formulae are the same, but it isn't. The stability of φ depends on how we partition its free variables into tuples. When we write $\varphi(x, y)$ we indicate a partition of the free variables in φ into two tuples x and y *in that order*. By $\varphi^*(y, x)$ we mean the same formula φ and indeed the same partition of the free variables into tuples x and y, but this time in the order y, x. (You might like to think of it as switching the main variable: x is the main variable of $\varphi(x, y)$, while y is the main variable of $\varphi^*(y, x)$. This way of viewing it will become useful when we start discussing φ- and φ^*-types.) To show that $\varphi^*(y, x)$ is unstable, we would need to find a_i, b_i for $i < \omega$ such that $\models \varphi^*(a_i, b_j)$ iff $i \leq j$, which is not the same as (1.1), since we have $\models \varphi^*(a_i, b_j)$ iff $\models \varphi(b_j, a_i)$, **not** $\models \varphi^*(a_i, b_j)$ iff $\models \varphi(a_i, b_j)$. Now, as (iii) states, we do in fact have that $\varphi(x, y)$ is stable iff $\varphi^*(y, x)$ is stable, but this requires proof.

Proofs of (i)–(iii). The proofs of (i) and (ii) are fairly straightforward, see [1, Remark 2.9] for details. Part (iii) follows from (vi) below by replacing ω with the reverse ordering ω^* ($m <^* n$ iff $n < m$, i.e. $0 > 1 > 2 > \cdots$). □

(iv) Suppose $a_i, b_i, i < \omega$ satisfy (1.1). Then the a_i are all distinct, as are all the b_i.

Proof. Suppose $a_l = a_k$. Without loss of generality assume $l < k$. Since $l \leq l$, we have $\models \varphi(a_l, b_l)$. Thus, since $a_l = a_k$, we have $\models \varphi(a_k, b_l)$. But $k \not\leq l$, a contradiction. Symmetrical reasoning covers the b_i. □

(v) A formula $\varphi(x, y)$ defines a relation $R := \{(a, b) : \models \varphi(a, b)\}$, which can be thought of as bipartite graph. Definition 1.1 then says that $\varphi(x, y)$ is unstable iff there exists an infinite bipartite subgraph of R of the following form (modulo a certain caveat):



Caveat. Although the a_i must all be distinct, as must all the b_i (see part (iv) above), the a_i and the b_i need not be distinct from each other; that is, there may exist $k, l < \omega$ such that $a_k = b_l$. For example, consider $(\mathbb{Z}, <)$ and let $\varphi(x, y)$ be the formula $x \leq y$. Let $a_i = b_i = i$. Then $\varphi(x, y)$ and a_i, b_i satisfy (1.1) but $a_i = b_i$ for all $i < \omega$. One can still think of the a_i and b_i as forming a bipartite graph though, just view the b_i as copies of the a_i.

(vi) In Definition 1.1, one can replace ω with any infinite linear ordering I; that is, there exist a_i, b_i for $i < \omega$ satisfying (1.1) if and only if there exist a'_i, b'_i for $i \in I$ satisfying (1.1).

Proof. ($\Rightarrow$) Suppose there exist a_i, b_i for $i < \omega$ satisfying (1.1) and let I be an infinite linear ordering. Let $\mathcal{L}_I := \mathcal{L} \cup \{a'_i, b'_i : i \in I\}$, where the a'_i, b'_i are constant symbols, and define T_I to be the $\mathcal{L}_I$-theory $T \cup \{\varphi(a'_i, b'_j) : i, j \in I \text{ and } i \leq j\}$ (recall that T is the ambient theory). Consider some finite $A \subset T_I$. Since any finite linear ordering can be embedded in ω, A has a model, namely some finite subset of the a_i, b_i satisfying (1.1). We can thus apply compactness to obtain a model of T_I and we are done.

($\Leftarrow$) The same compactness argument works again, just switch ω and I. □

Lemma 1.2. *Let $\varphi(x, y)$ be a stable $\mathcal{L}$-formula, let $\mathcal{M}$ be a model and let $c \in \bar{\mathcal{M}}$. Then there exists a formula $\psi(y)$ over $\mathcal{M}$ (i.e. $\psi(y) \in \mathcal{L}_\mathcal{M}$) which is a positive Boolean combination of formulae $\varphi(c', y)$ for some $c' \in \mathcal{M}$ such that*

$$\models \varphi(c, b) \text{ if and only if } \models \psi(b)$$

for all $b \in \mathcal{M}$. Moreover, if $\mathcal{M}$ is $\aleph_0$-saturated, then we can choose the c' to be realisations of tp(c).[4]

Externally and internally definable sets

This lemma roughly says that, in a stable setting, externally definable sets are internally definable; more precisely, it says that if $\varphi(x, y)$ is stable, then φ-types are definable, in a special way (see Comment 1.B(ii)).

Proof of Lemma 1.2. By induction we will find $c_i, a_i, b_i \in \mathcal{M}$ for $i < \omega$ with the following properties:

(1) $\models \varphi(c, a_i) \wedge \neg\varphi(c, b_i)$ for all i;

(2) $\models \varphi(c_j, a_i) \wedge \neg\varphi(c_j, b_i)$ for all $i \leq j$; and

(3) $\models \varphi(c_j, a_i) \rightarrow \varphi(c_j, b_i)$ for all $j < i$.

We will see that whenever the inductive construction fails, the lemma is in fact true and so we will push on and assume that the construction can continue. We will then eventually see that the existence of the c_i, a_i, b_i contradicts the stability of $\varphi(x, y)$ and we will be done.

Let's start the induction. First consider the base case $i = 0$. We want to find $c_0, a_0, b_0 \in \mathcal{M}$ such that

(1)′ $\models \varphi(c, a_0) \wedge \neg\varphi(c, b_0)$;

(2)′ $\models \varphi(c_0, a_0) \wedge \neg\varphi(c_0, b_0)$; and

(3)′ $\models \varphi(c_j, a_0) \rightarrow \varphi(c_j, b_0)$ for all $j < 0$.

(3)′ holds vacuously. (2)′ follows from (1)′ by Tarski–Vaught: Suppose we have $a_0, b_0 \in \mathcal{M}$ that satisfy (1)′. Let $\chi(x)$ be $\varphi(x, a_0) \wedge \neg\varphi(x, b_0)$. Notice that $\chi(x)$ is over $\mathcal{M}$. Thus, since $\models \chi(c)$ (by (1)′) and $\mathcal{M} \prec \bar{\mathcal{M}}$, by Tarski–Vaught there exists $c_0 \in \mathcal{M}$ such that $\mathcal{M} \models \chi(c_0)$ and so (2) holds.[5]

We are left to prove (1)′. Suppose that there are no $a_0, b_0 \in \mathcal{M}$ satisfying (1)′. Then either $\mathcal{M} \models \forall y\, \varphi(c, y)$ or $\mathcal{M} \models \forall y\, \neg\varphi(c, y)$. First suppose $\mathcal{M} \models \forall y\, \varphi(c, y)$. By Tarski–Vaught we can find $c' \in \mathcal{M}$ such that $\mathcal{M} \models \forall y\, \varphi(c', y)$. Let $\psi(y)$ be $\varphi(c', y)$. Then $\models \varphi(c', b)$ and $\models \psi(b)$ for all $b \in \mathcal{M}$ and thus the lemma is true. Now suppose $\mathcal{M} \models \forall y\, \neg\varphi(c, y)$. Again using Tarski–Vaught, we can find $c' \in \mathcal{M}$ such that $\mathcal{M} \models \forall y\, \neg\varphi(c', y)$. Let $\psi(y)$ be $\varphi(c', y)$. Then $\not\models \varphi(c', b)$ and $\not\models \psi(b)$ for all $b \in \mathcal{M}$ and thus the lemma is true. So if we cannot find $a_0, b_0 \in \mathcal{M}$ satisfying (1)′, then the lemma holds, so assume that such a_0, b_0 exist.

We now come to the induction step. Suppose that we already have c_i, a_i, b_i satisfying (1)–(3) for all $i < n$ (where $n > 0$). We will find suitable c_n, a_n, b_n.

Claim 1. We may assume that there exist $a_n, b_n \in \mathcal{M}$ such that $\models \varphi(c, a_n) \wedge \neg\varphi(c, b_n)$ and $\models \varphi(c_j, a_n) \rightarrow \varphi(c_j, b_n)$ for all $j < n$.

Proof of Claim 1. Suppose that no such a_n, b_n exist. Then for all $a, b \in \mathcal{M}$ such that $\models \bigwedge_{j<n} \varphi(c_j, a) \rightarrow \varphi(c_j, b)$ we have

$$\models \varphi(c, a) \rightarrow \varphi(c, b). \tag{$*$}$$

For each $a \in \mathcal{M}$ such that $\models \varphi(c, a)$,[6] let

$$J_a := \{j < n : \models \varphi(c_j, a)\}.$$

Each J_a is a subset of $\{0, 1, \ldots, n-1\}$ and so there are only finitely many J_a. Let $\psi(y)$ be the formula

$$\bigvee_{\substack{a \in \mathcal{M} \text{ s.t.} \\ \models \varphi(c,a)}} \bigwedge_{j \in J_a} \varphi(c_j, y).$$

This formula is essentially finite (and thus legitimate) as there are only finitely many J_a. Notice that $\psi(y) \in \mathcal{L}_{\mathcal{M}}$.

Subclaim. For all $b \in \mathcal{M}$ we have $\models \varphi(c, b)$ if and only if $\models \psi(b)$.

Proof of subclaim. ($\Rightarrow$) Suppose $\models \varphi(c, b)$ for $b \in \mathcal{M}$. Then $\models \bigwedge_{j \in J_b} \varphi(c_j, b)$ and so $\models \psi(b)$. (Note that the empty conjunction is always true and so the case $J_b = \varnothing$ is okay.)

[4] For $b \in \bar{\mathcal{M}}$ and $A \subseteq \bar{\mathcal{M}}$, $\mathrm{tp}(b/A)$ denotes the complete type of b over A in $\bar{\mathcal{M}}$, i.e. $\mathrm{tp}(b/A) = \mathrm{tp}^{\bar{\mathcal{M}}}(b/A) := \{\chi(x) \in \mathcal{L}_A : \models \chi(b)\}$. If $A = \varnothing$, then we often simply write $\mathrm{tp}(b)$.

[5] We will use Tarski–Vaught in this way fairly often. We won't go into as much detail when we use it again.

[6] Note that such an $a \in \mathcal{M}$ exists, e.g. a_0.

($\Leftarrow$) Suppose that for some $a \in \mathcal{M}$ such that $\models \varphi(c,a)$ we have $\bigwedge_{j \in J_a} \varphi(c_j, b)$. Then $\models \varphi(c_j, b)$ whenever $\models \varphi(c_j, a)$ (by the definition of J_a) and so $\models \bigwedge_{j<n} \varphi(c_j, a) \to \varphi(c_j, b)$. Thus $\models \varphi(c, b)$ by $(*)$. □

The subclaim shows that if no such a_n, b_n exist, then the lemma is true. We may thus assume that such a_n, b_n do exist and so Claim 1 is proved. □

We are now left to find a suitable $c_n \in \mathcal{M}$, namely one such that $\models \varphi(c_n, a_i) \wedge \neg\varphi(c_n, b_i)$ for all $i \le n$. Let $\chi(z)$ be the formula

$$\bigwedge_{i \le n} \varphi(z, a_i) \wedge \bigwedge_{i \le n} \neg\varphi(z, b_i).$$

Since $\chi(x)$ is over $\mathcal{M}$ and $\models \chi(c)$, by Tarski–Vaught there exists $c_n \in \mathcal{M}$ such that $\mathcal{M} \models \chi(c_n)$. So the construction is finished.

Ramsey's Theorem

We now apply Ramsey's Theorem in a very useful way:

Claim 2. Without loss of generality we may assume either

(i) $\models \neg\varphi(c_j, a_i)$ for all $j < i$; or

(ii) $\models \varphi(c_j, a_i)$ for all $j < i$.

Proof of Claim 2. We construct a colouring of 2-element subsets of ω as follows: for $j < i$, colour $\{j, i\}$ red iff $\models \neg\varphi(c_j, a_i)$ or blue iff $\models \varphi(c_j, a_i)$. By Ramsey's Theorem, there exists an infinite subset $A \subseteq \omega$ such that all 2-element subsets of A are monochromatic. Define $f\colon \omega \to A$ by $f(n) := \min(A \setminus \{f(i) : i < n\})$. Then $c_{f(i)}, a_{f(i)}, b_{f(i)}$ satisfy (1)–(3) and either (i) or (ii) and so we are done. □

In case (i) we have $\models \varphi(c_j, a_i)$ iff $i \le j$ by (2), contradicting the stability of $\varphi(x, y)$ (cf. Comment 1.A(iii)). In case (ii) we have $\models \varphi(c_j, b_i)$ iff $j < i$ by (2) and (3), also contradicting the stability of $\varphi(x, y)$ (cf. Comment 1.A(i)). So both cases lead to contradiction and so the main part of the lemma is proved.

Finally, for the moreover clause of the lemma, notice that we can choose the c_j to realise $\operatorname{tp}(c)$ if $\mathcal{M}$ is $\aleph_0$-saturated. □

φ-formulae and φ-types

Definition 1.3. Fix some formula $\varphi(x, y) \in \mathcal{L}$.

(i) A *φ-formula over a set A* is a formula $\chi(x) \in \mathcal{L}_A$ such that $\chi(x)$ is equivalent (w.r.t. the ambient theory T) to a Boolean combination of formulae $\varphi(x, b)$ for some $b \in \bar{\mathcal{M}}$.[7]

(ii) A *complete φ-type over A* is a maximally consistent (w.r.t. the ambient theory T) collection of φ-formulae over A. The set of all complete φ-types over A is denoted $S_\varphi(A)$. For a given $b \in \bar{\mathcal{M}}$, $\operatorname{tp}_\varphi(b/A)$ denotes the complete φ-type of all φ-formulae over A realised by b; that is,

$$\operatorname{tp}_\varphi(b/A) := \{\chi(x) : \chi(x) \text{ is a } \varphi\text{-formulae over } A \text{ s.t.} \models \chi(b)\}.$$

Important terminological note

Unless otherwise specified, *φ-type* will mean *complete φ-type*. We will often just write *type* for *φ-type* when the context is clear, but do be mindful of the difference between types and φ-types, especially in Lemma 1.6 below. We will sometimes write *standard type* to contrast with *φ-type*.

Comment 1.B.

(i) If $\mathcal{M} \prec \bar{\mathcal{M}}$, then a φ-formula over $\mathcal{M}$ is equivalent to a Boolean combination of formulae $\varphi(x, b)$ for some $b \in \mathcal{M}$ (by Tarski–Vaught) and thus a complete φ-type over $\mathcal{M}$ is essentially a choice of either $\varphi(x, b)$ or $\neg\varphi(x, b)$ for each $b \in \mathcal{M}$.

[7] What we refer to as a *φ-formula* is sometimes referred to as a *generalised φ-formula*, e.g. [1, Definition 6.9], since we allow the b to be tuples in $\bar{\mathcal{M}}$, rather than restricting them to A.

(ii) If $\varphi(x,y)$ is stable, then any $p(x) \in S_\varphi(\mathcal{M})$ is *definable*, i.e. there exists a formula $\psi(y)$ over $\mathcal{M}$ such that for all $b \in \mathcal{M}$, $\varphi(x,b) \in p(x)$ iff $\mathcal{M} \models \psi(b)$; we call $\psi(y)$ the *φ-definition* of $p(x)$ or just the *definition* of $p(x)$ if the context is clear.[8] Moreover, $\psi(y)$ is a positive Boolean combination of formulae $\varphi(a,y)$ for some $a \in \mathcal{M}$ and thus $\psi(y)$ is a φ^*-formula (since $\varphi(a,y)$ is $\varphi^*(y,a)$). *Definable φ-types*

Proof. Since $\bar{\mathcal{M}}$ is saturated, there exists $c \in \bar{\mathcal{M}}$ realising $p(x)$. The rest follows from Lemma 1.2 and part (i). □

Lemma 1.4. *Let T be a countable complete theory and let $\varphi(x,y) \in \mathcal{L}$. Then the following are equivalent:* *Counting types*

(i) $\varphi(x,y)$ is stable.

(ii) Every $p(x) \in S_\varphi(\mathcal{M})$ is definable for every $\mathcal{M} \models T$.

(iii) For all cardinals $\lambda \geq \aleph_0$ and $|A| \leq \lambda$, $|S_\varphi(A)| \leq \lambda$.

(iv) For all countable A, $|S_\varphi(A)|$ is countable.

Proof. (i) ⇒ (ii) By Lemma 1.2.

(ii) ⇒ (iii) Let $|A|, \aleph_0 \leq \lambda$. By the Downward Löwenheim–Skolem Theorem, there exists $\mathcal{M}$ such that $A \subseteq \mathcal{M}$ and $|\mathcal{M}| \leq \lambda$. (Notice that $\mathcal{L}$ must be countable because T is countable.) Any type $p \in S_\varphi(A)$ extends to a type $p' \in S_\varphi(\mathcal{M})$, so we may assume $A = \mathcal{M}$. Now, there are $|\mathcal{M}|$-many formulae $\psi(y)$ over $\mathcal{M}$ and thus there are $|\mathcal{M}|$-many possible candidates for the definition of $p \in S_\varphi(\mathcal{M})$. Therefore $|S_\varphi(A)| \leq \lambda$ by (ii).

(iii) ⇒ (iv) Immediate.

¬(i) ⇒ ¬(iv) Suppose that $\varphi(x,y)$ is unstable. By applying Comment 1.A(vi) with $I = \mathbb{R}$ and then restricting the b_i to $i \in \mathbb{Q}$, we can find a_r for $r \in \mathbb{R}$ and b_q for $q \in \mathbb{Q}$ such that $\models \varphi(a_r, b_q)$ iff $r \leq q$. Let $A = \{b_q : q \in \mathbb{Q}\}$. Then the a_r realise continuum-many distinct types $p(x) \in S_\varphi(A)$. □

Before we move on to more results regarding local stability, we need to go over a few background topics:

Cantor–Bendixson rank. Let X be a compact topological space and let $A \subseteq X$ be any subset. Define A' to be the set of all limit points of A.[9] Notice that for the whole topological space X, X' is X minus all the isolated points of X. We define the *Cantor–Bendixson derivative* of X inductively as follows: *CB-rank*

(1) $X^{(0)} = X$.

(2) $X^{(\alpha+1)} = \left(X^{(\alpha)}\right)'$.

(3) $X^{(\alpha)} = \bigcap_{i<\alpha} X^{(i)}$ if α is a limit ordinal.

(4) $X^{(\infty)} = \bigcap_{\alpha \in On} X^{(\alpha)}$, where On is the class of all ordinals.

Note that each $X^{(\alpha)}$ is closed and that $X^{(0)} \supseteq X^{(1)} \supseteq \cdots \supseteq X^{(\alpha)} \supseteq X^{(\alpha+1)} \supseteq \cdots \supseteq X^{(\infty)}$.

For a point $x \in X$, the *Cantor–Bendixson rank* (or just *CB-rank*) of x is defined to be the largest ordinal α such that $x \in X^{(\alpha)}$, or ∞ if $x \in X^{(\infty)}$. The CB-rank of a non-empty closed subset $V \subseteq X$ is defined to be the maximal CB-rank of all the points in V,[10] which is well-defined by the comment below. The CB-rank of $\varnothing$ is defined to be -1. The CB-ranks of x and V are denoted $\mathrm{CB}_X(x)$ and $\mathrm{CB}_X(V)$ respectively.

[8] We say *the* φ-definition of $p(x)$ because $\psi(y)$ is unique up to equivalence: Suppose that $\psi'(y)$ is another φ-definition of $p(x)$. Then for all $b \in \mathcal{M}$ we have $\mathcal{M} \models \psi(b)$ iff $\varphi(x,b) \in p(x)$ iff $\mathcal{M} \models \psi'(b)$, i.e. $\mathcal{M} \models \psi(b)$ iff $\mathcal{M} \models \psi'(b)$ for all $b \in \mathcal{M}$.

[9] A *limit point* of a subset A is a point $x \in X$ such that for all neighbourhoods U of x, $(A \cap U) \setminus \{x\} \neq \varnothing$. A point $a \in A$ that is not a limit point of A is called an *isolated point of* A or said to be *isolated in* A.

[10] In detail: If there exists $v \in V$ such that $CB_X(v) = \infty$, then the CB-rank of V is defined to be ∞. If no such v exists, then the CB-rank of V is defined to be the largest ordinal α for which there exists $v \in V$ with $CB_X(v) = \alpha$.

Comment 1.C. Let X be a compact space.

(i) $\mathrm{CB}_X(V)$ is well-defined for all closed $V \subseteq X$.

(ii) Let V be closed and suppose $CB_X(V) = \alpha < \infty$. Then the subset $V_0 := \{v \in V : \mathrm{CB}_X(v) = \alpha\}$ is non-empty and finite.

(iii) For all $x \in X$, $\mathrm{CB}_X(x) = \alpha$ iff x is isolated in the set $\{y \in X : \mathrm{CB}_X(y) \geq \alpha\}$ and $\mathrm{CB}_X(x) = \infty$ iff x is not isolated in the set $\{y \in X : \mathrm{CB}_X(y) = \infty\}$.

Proof. (i) It suffices to show the following for any limit ordinal α: If for every $\beta < \alpha$ there exists $v \in V$ with $\mathrm{CB}_X(v) \geq \beta$, then there exists some $v_0 \in V$ with $\mathrm{CB}_X(v_0) \geq \alpha$. This follows from the compactness of X: since the $V \cap X^{(\beta)}$ are closed and nested (i.e. $V \cap X^{(\beta)} \supseteq V \cap X^{(\beta+1)}$ for each $\beta < \alpha$), $V \cap X^{(\alpha)} = \bigcap_{\beta<\alpha} V \cap X^{(\beta)}$ is non-empty.

(ii) V_0 is non-empty by definition. Now, by way of contradiction, suppose that $V_0 \cap V_0' \neq \varnothing$. Let $x \in V_0 \cap V_0'$. Then x is a limit point of V_0 and thus also a limit point of $X^{(\alpha)}$, since $V_0 \subseteq X^{(\alpha)}$. So $x \in X^{(\alpha+1)}$, i.e. $\mathrm{CB}_X(x) \geq \alpha + 1$. But $x \in V_0$ and so $\mathrm{CB}_X(x) = \alpha$, a contradiction. Therefore $V_0 \cap V_0' = \varnothing$, i.e. no point in V_0 is a limit point of V_0, and thus for each $x \in V_0$ there exists an open set U_x such that $V_0 \cap U_x = \{x\}$. Now, $V_0 = V \cap X^{(\alpha)}$ is closed and thus compact (under the subspace topology). So, since $\{V_0 \cap U_x\}_{x \in V_0}$ is an open cover of V_0, it has a finite subcover. But each $V_0 \cap U_x$ is a singleton, hence V_0 is finite.

(iii) This follows from the fact that $\{y \in X : \mathrm{CB}_X(y) \geq \alpha\} = X^{(\alpha)}$ and $\{y \in X : \mathrm{CB}_X(y) = \infty\} = X^{(\infty)}$. □

Stone space

Stone space. We can endow $S_\varphi(A)$ with a topology. For each φ-formula $\chi(x)$ over A, define

$$U_\chi := \{p(x) \in S_\varphi(A) : \chi(x) \in p(x)\}.$$

We then define the topology by taking these sets U_χ as the basic open sets.[11] Equipped with this topology, $S_\varphi(A)$ is called the *Stone space of φ over A*. Note that $S_\varphi(A)$ is a totally disconnected, compact, Hausdorff space under this topology and that in a saturated model $\mathcal{M}$, the Morley rank of a φ-type p is equal to the CB-rank of p in $S_\varphi(\mathcal{M})$.

Logical vs topological compactness

Interlude. Let T be a first-order, 1-sorted theory. The type space $S_n(T)$ is the space of ultrafilters on $B_n(T)$, where $B_n(T)$ is the Boolean algebra of formulae $\varphi(x_1, \ldots, x_n)$ up to equivalence modulo T. $S_n(T)$ is Hausdorff, compact, 0-dimensional and totally disconneted and thus a profinite space. The logical compactness of first-order logic is equivalent to the topological compactness of $S_n(T)$. End of interlude.

Trees

Tree notation. Let α, β be ordinals. We define ${}^\alpha\beta$ to be the set of all functions from α to β. We then define ${}^{<\alpha}\beta$ as follows:[12]

$${}^{<\alpha}\beta := \bigcup_{\gamma<\alpha} {}^\gamma\beta$$

Let's consider the case where $\alpha = \omega$ and $\beta = 2$. A function $\lambda \in {}^\omega 2$ is simply an infinite sequence of 0s and 1s and thus we can think of ${}^\omega 2$ as the complete binary tree,[13] where the empty function is the root and each $\lambda \in {}^\omega 2$ is a particular infinite branch down the tree.[14] Each node in the tree is given by some $\mu \in {}^{<\omega}2$, since μ is a *finite* sequence of 0s and 1s. If $\mu \in {}^{<\omega}2$ is a sequence of length n, we write $\mu^\frown(i)$ for the $(n+1)$-length sequence extending μ by i,[15] so $\mu^\frown(0)$ and $\mu^\frown(1)$ are the nodes immediately below μ. Nodes and finite branches are equivalent, since for each $\mu \in {}^{<\omega}2$, $\mu = \lambda \restriction n$ for some $\lambda \in {}^\omega 2$ and $n < \omega$ (and vice versa). Notice that we can also view ${}^{<\omega}2$ as the

[11] Equivalently, we could take the basic open sets to be $V_\chi := \{p(x) \in S_\varphi(A) : p(x) \cup \{\chi(x)\}$ is consistent$\}$, since φ-types in $S_\varphi(A)$ are *complete* types and thus $p(x) \cup \{\chi(x)\}$ is consistent iff $\chi(x) \in p(x)$, hence $V_\chi = U_\chi$.

[12] Note that there is quite some variety in notation in the literature: ${}^\alpha\beta = \beta^\alpha$ and ${}^{<\alpha}\beta = {}^{\alpha>}\beta = \beta^{<\alpha}$.

[13] By *the complete binary tree* we mean the (isomorphism class) of the infinite graph formed by starting at a root node and then bisecting ad infinitum.

[14] By a *branch* we mean a path starting at the root which contains each node at most once, i.e. it doesn't go back on itself. We say that a branch goes *down* the tree, but this use of direction is solely to help intuition; *up*, *along* or any other suitable preposition would do just as well.

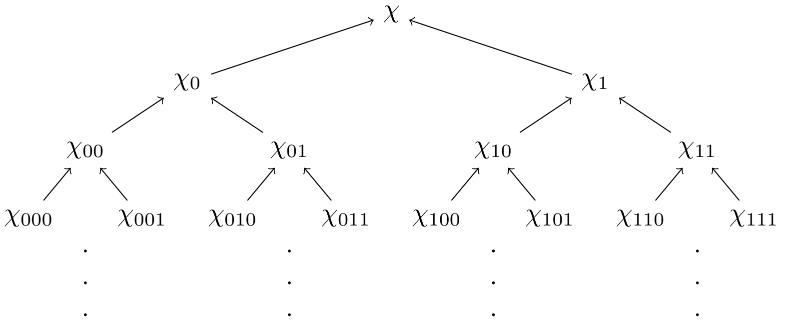
[15] Again, there are different notations: $\mu^\frown(i) = \mu^\wedge(i) = \mu^\frown\langle i\rangle = \mu^\wedge\langle i\rangle = \mu^\frown i = \mu^\wedge i$.

complete binary tree, the distinction being that ${}^{<\omega}2$ contains finite functions $n \to 2$ for arbitrarily large n, while ${}^{\omega}2$ contains infinite functions $\omega \to 2$. This is a subtle but important difference: ${}^{<\omega}2$ is countable while ${}^{\omega}2$ has cardinality of the continuum. That might sound contradictory at first, but it isn't: ${}^{<\omega}2$ is the set of nodes, while ${}^{\omega}2$ is the set of infinite branches.

Corollary 1.5. *Let $X = S_\varphi(\mathcal{M})$. If $\varphi(x, y)$ is stable, then $\mathrm{CB}_X(p) < \infty$ for every $p \in S_\varphi(\mathcal{M})$.*[16]

Proof. By way of contradiction, suppose $\mathrm{CB}_X(p) = \infty$ for some $p \in S_\varphi(\mathcal{M})$. We will find continuum-many φ-types over a countable set, thereby contradicting Lemma 1.4(iv).

Let $\mathrm{CB}_X(p) = \infty$ and let $\chi \in p$. By Comment 1.C(iii), p is not isolated in $Z := \{y \in X : \mathrm{CB}_X(y) = \infty\}$, i.e. any open set containing p intersects Z in at least one point not equal to p. Thus, since $p \in U_\chi$, there exists $q \in (Z \cap U_\chi) \setminus \{p\}$. So $\chi \in q$, $q \neq p$ and $\mathrm{CB}_X(q) = \infty$. Since $q \neq p$, there exists a φ-formula χ_0 such that $\chi_0 \in p$ and $\neg\chi_0 \in q$. But χ is in both p and q and thus each of χ_0 and $\neg\chi_0$ implies χ (if either implied $\neg\chi$, then we would have a contradiction by modus ponens). Let χ_1 be $\neg\chi_0$. Since $\mathrm{CB}_X(p) = \mathrm{CB}_X(q) = \infty$, we can repeat this argument indefinitely to obtain a complete binary tree of φ-formulae



such that each branch is consistent but any two distinct branches are inconsistent. More formally, for each $\mu \in {}^{<\omega}2$ we apply the above argument to find $\chi_{\mu^\frown(0)}$ and $\chi_{\mu^\frown(1)}$ such that $\chi_{\mu^\frown(1)}$ is $\neg\chi_{\mu^\frown(0)}$, $\chi_{\mu^\frown(0)} \to \chi_\mu$ and $\chi_{\mu^\frown(1)} \to \chi_\mu$. This gives us a binary tree of formulae $\{\chi_\mu : \mu \in {}^{<\omega}2\}$ such that $\Sigma_\lambda := \{\chi_{\lambda\restriction n} : n < \omega\}$ is consistent for each $\lambda \in {}^{\omega}2$ but $\{\Sigma_\lambda : \lambda \in {}^{\omega}2\}$ is pairwise inconsistent. Let A be the set of all parameters (= elements of $\mathcal{M}$) which appear in formulae in $\{\chi_\mu : \mu \in {}^{<\omega}2\}$. Notice that A is countable, since ${}^{<\omega}2$ is countable and each χ_μ contains only finitely many parameters. Now, each Σ_λ is a partial φ-type over A and so is contained in some complete φ-type over A. But $\{\Sigma_\lambda : \lambda \in {}^{\omega}2\}$ is pairwise inconsistent and thus these complete φ-types must all be distinct. Hence we have ${}^{\omega}2$-many φ-types over A, which is the contradiction we were looking for. ☐

Algebraic closure, automorpisms and canonical parameters. Before we apply the corollary to obtain some more results, we need to go over some more background material. For $A \subseteq \bar{\mathcal{M}}$, $\mathrm{acl}^{\mathrm{eq}}(A)$ is the algebraic closure of A taken in $\bar{\mathcal{M}}^{\mathrm{eq}}$ (rather than in $\bar{\mathcal{M}}$). In detail: We say that $b \in B$ is *algebraic over A in $\mathcal{M}$* iff there exists a formula $\theta(x) \in \mathcal{L}_A$ such that $\mathcal{M} \models \theta(b)$ and $\theta(\mathcal{M}) := \{a \in \mathcal{M} : \mathcal{M} \models \theta(a)\}$ is finite. We then define $\mathrm{acl}_{\mathcal{M}}(A) := \{b \in \mathcal{M} : b \text{ is algebraic over } A \text{ in } \mathcal{M}\}$. We then further define $\mathrm{acl}^{\mathrm{eq}}(A) := \mathrm{acl}_{\bar{\mathcal{M}}^{\mathrm{eq}}}(A)$. $\mathrm{acl}^{\mathrm{eq}}$

The set of automorpisms of a model $\mathcal{M}$ is denoted $\mathrm{Aut}(\mathcal{M})$. For a set A, we define $\mathrm{Aut}(\mathcal{M}/A)$ to be the set of automorphisms of $\mathcal{M}$ that fix A pointwise, i.e. $\sigma \in \mathrm{Aut}(\mathcal{M}/A)$ iff $\sigma \in \mathrm{Aut}(\mathcal{M})$ and $\sigma(a) = a$ for all $a \in A$. An automorphism acts on formulae in the following way: For $\varphi(x, y) \in \mathcal{L}$, $b \in \mathcal{M}$ and $\sigma \in \mathrm{Aut}(\mathcal{M})$, we define $\varphi(x, b)^\sigma := \varphi(x, \sigma(b))$. We can extend this to φ-types: for $p(x) \in S_\varphi(A)$, where $A \subseteq \mathcal{M}$, we define $p(x)^\sigma := \{\chi(x)^\sigma : \chi(x) \in p(x)\}$. Thus an automorphism $\sigma \in \mathrm{Aut}(\mathcal{M})$ induces a map $\sigma\colon S_\varphi(\mathcal{M}) \to S_\varphi(\mathcal{M})$. $\mathrm{Aut}(\mathcal{M}/A)$

Let $X \subseteq \mathcal{M}$ be a definable set, i.e. $X = \varphi(\mathcal{M}, a)$ for some $\varphi(x, y) \in \mathcal{L}$ and $a \in \mathcal{M}$, where $\varphi(\mathcal{M}, a) := \{b \in \mathcal{M} : \mathcal{M} \models \varphi(b, a)\}$. We can define an equivalence relation E_φ on $\mathcal{M}$ as follows: $E_\varphi(a_1, a_2)$ iff $\mathcal{M} \models \forall x\, \varphi(x, a_1) \leftrightarrow \varphi(x, a_2)$. So, in other words, $E_\varphi(a_1, a_2)$ is true iff $\varphi(x, a_1)$ and $\varphi(x, a_2)$ both define X. Let a/E_φ denote the equivalence class of a under E_φ. We call a/E_φ the *canonical parameter* of X. X is definable over a/E_φ in $\mathcal{M}^{\mathrm{eq}}$. *Canonical parameters*

[16] A topological space X with this property ($\mathrm{CB}_X(x) < \infty$ for every $x \in X$) is called *scattered*.

Comment 1.D. Let $A \subseteq \bar{\mathcal{M}}$ and consider a definable set $X \subseteq \bar{\mathcal{M}}$ with canonical parameter e.[17]

(i) For all $\sigma \in \mathrm{Aut}(\bar{\mathcal{M}})$, $\sigma(X) = X$ if and only if $\sigma(e) = e$.

(ii) $\sigma(X) = X$ for all $\sigma \in \mathrm{Aut}(\bar{\mathcal{M}}/A)$ if and only if X is defined over A. In such a situation we say that X is *over* A.

(iii) X has finitely many images under $\mathrm{Aut}(\bar{\mathcal{M}}/A)$ if and only if X is defined over $\mathrm{acl}^{\mathrm{eq}}(A)$. In such a situation we say that X is *almost over* A.

Proof. Let $X := \varphi(\bar{\mathcal{M}}, a)$.

(i) Consider some $\sigma \in \mathrm{Aut}(\bar{\mathcal{M}})$.

($\Leftarrow$) Suppose that $\sigma(e) = e$, i.e. $\sigma(b) \in e$ iff $b \in e$ (σ is a bijection). First let $c \in X$. Then $\models \varphi(c, a)$. Thus, since σ is an automorphism, $\models \varphi(\sigma(c), \sigma(a))$. But $\sigma(a) \in e$ and so $\varphi(x, \sigma(a))$ also defines X. Thus $\sigma(c) \in X$. Hence $\sigma(X) \subseteq X$.

Now let $c \in \sigma(X)$. Then $c = \sigma(d)$ for some $d \in X$. Thus $\models \varphi(d, a)$ and so $\models \varphi(\sigma(d), \sigma(a))$. But $\sigma(a) \in e$ and so $c = \sigma(d) \in X$. Hence $X \subseteq \sigma(X)$ and we are done.

($\Rightarrow$) Now suppose that $\sigma(X) = X$. We want to show that $\sigma(b) \in e$ iff $b \in e$. First let $b \in e$. Then:

$$\begin{array}{rcll} c \in \sigma(X) & \iff & \sigma^{-1}(c) \in X & (\sigma \text{ is a bijection}) \\ & \iff & \models \varphi(\sigma^{-1}(c), b) & (b \in e) \\ & \iff & \models \varphi(c, \sigma(b)) & (\sigma \text{ is an automorphism}) \end{array}$$

So $\varphi(x, \sigma(b))$ defines $\sigma(X)$. But $\sigma(X) = X$ and so $\varphi(x, \sigma(b))$ also define X, i.e. $\sigma(b) \in e$.

Now let $\sigma(b) \in e$. Then

$$\begin{array}{rcll} c \in X & \iff & \sigma(c) \in \sigma(X) & (\sigma \text{ is a bijection}) \\ & \iff & \sigma(c) \in X & (\sigma(X) = X) \\ & \iff & \models \varphi(\sigma(c), \sigma(b)) & (\sigma(b) \in e) \\ & \iff & \models \varphi(c, b) & (\sigma \text{ is an automorphism}) \end{array}$$

So $b \in e$.

(ii) ($\Leftarrow$) This is very similar to the proof of the ($\Leftarrow$) direction of part (i) and is left as an exercise.

($\Rightarrow$) Suppose that $\sigma(X) = X$ for all $\sigma \in \mathrm{Aut}(\bar{\mathcal{M}}/A)$.

Let $p(y) := \mathrm{tp}(a/A)$. We claim that $b \in e$ for all $b \models p$.[18] So suppose that $b \models p$. Then $\mathrm{tp}(b/A) = \mathrm{tp}(a/A)$. Thus, since $\bar{\mathcal{M}}$ is saturated, there exists $\sigma \in \mathrm{Aut}(\bar{\mathcal{M}}/A)$ such that $\sigma(a) = b$. Thus $b \in e$ by part (i) and the claim is proved.

So $b \models p$ implies $b \in e$. Thus by compactness[19] there exists a formula $\psi(y) \in p(y)$ such that $\models \psi(b)$ implies $b \in e$. Then the $\mathcal{L}_A$-formula $\exists y\,(\psi(y) \wedge \varphi(x, y))$ defines X and we are done.

(iii) The proof is along the same lines as that of part (ii) and is left as an exercise. □

We are now ready to apply Corollary 1.5:

Existence **Lemma 1.6.** *Let $\varphi(x, y)$ be stable and $p(x) \in S(A)$,[20] where $A \subseteq \mathcal{M}$. Then there exists $q(x) \in S_\varphi(\mathcal{M})$ such that*

(i) $p(x) \cup q(x)$ is consistent and

(ii) $q(x)$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$, i.e. the φ-definition of $q(x)$ is over $\mathrm{acl}^{\mathrm{eq}}(A)$.

[17] Working in $\bar{\mathcal{M}}$ is simply for convenience, the results hold for any saturated $\mathcal{M}$. Indeed, part (i) in fact holds for any $\mathcal{M}$, not just saturated $\mathcal{M}$.

[18] We write $b \models p$ to mean that b realises the type p.

[19] We use compactness in the following way: Let $\Sigma(y)$ be a partial type such that for all $b \in \bar{\mathcal{M}}$, $b \models \Sigma$ implies $\psi(b)$. Then there exists a finite $\Sigma' \subseteq \Sigma$ such that for all $b \in \bar{\mathcal{M}}$, $\models \bigwedge \Sigma'(b)$ implies $\models \psi(b)$.

Proof. We may assume that $\mathcal{M}$ is saturated. (If not, pass to $\bar{\mathcal{M}}$, find q and then restrict q to $\mathcal{M}$; $q \upharpoonright \mathcal{M}$ will have the same desired properties.) We thus have Comment 1.D at our disposal.

Let Y be the subset of $S_\varphi(\mathcal{M})$ consisting of all types consistent with $p(x)$. Y is closed, since any $r(x) \notin Y$ is contained in an open set U_χ for some φ-formula over A such that $\chi(x) \notin p(x)$, and $U_\chi \cap Y = \varnothing$, as otherwise $\chi(x)$ would be consistent with $p(x)$, a contradiction. By Corollary 1.5 and Comment 1.C(ii), the subset $Y_0 \subseteq Y$ consisting of all types of maximal CB-rank in Y is non-empty and finite. Let $q(x) \in Y_0$. Then, since $q(x) \in Y$, $p(x)$ and $q(x)$ are consistent. We are left to show that $q(x)$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$.

Consider some $\sigma \in \mathrm{Aut}(\mathcal{M}/A)$. Since the formulae in $p(x)$ are $\mathcal{L}_A$-formulae, $p(x)^\sigma = p(x)$. Y is thus setwise invariant under $\mathrm{Aut}(\mathcal{M}/A)$, as the image of any type containing $p(x)$ will still contain $p(x)$. Thus Y_0 is also setwise invariant under $\mathrm{Aut}(\mathcal{M}/A)$ by "English grammar".[21] So $\{q^\sigma : \sigma \in \mathrm{Aut}(\mathcal{M}/A)\}$ is finite, since q can be mapped only to other members of Y_0 and Y_0 is finite. Now, by Comment 1.B(ii), $q(x)$ is definable by some formula $\psi(y) \in \mathcal{L}_{\mathcal{M}}$. Thus $\{\sigma(\psi(\mathcal{M})) : \sigma \in \mathrm{Aut}(\mathcal{M}/A)\}$ is also finite, since the two sets are in bijection (exercise). Therefore $\psi(y)$ is over $\mathrm{acl}^{\mathrm{eq}}(A)$ by Comment 1.D(iii). □

Note. One can still apply this lemma to φ-types, not just standard types: Let $r(x) \in S_\varphi(A)$. Consider some extension $p(x) \in S(A)$ of $r(x)$, i.e. $r(x) \subseteq p(x)$. Such a $p(x)$ exists, although it may not be unique, since every φ-type extends to *some* standard type. Now apply the lemma to $p(x)$ to obtain $q(x) \in S_\varphi(\mathcal{M})$; $q(x)$ is an extension of $r(x)$. Note that $q(x)$ *is* unique (see Lemma 1.8 below).

Notation. In light of Lemma 1.6 and for the sake of brevity, we will henceforth write $\mathrm{acl}(A)$ for $\mathrm{acl}^{\mathrm{eq}}(A)$. *Important notational note*

Lemma 1.7. *Suppose that $\varphi(x, y)$ is stable, $p(x) \in S_\varphi(\mathcal{M})$ and $q(y) \in S_{\varphi^*}(\mathcal{M})$. Let $\psi(y), \chi(x) \in \mathcal{L}_{\mathcal{M}}$ be the φ-definition of $p(x)$ and the φ^*-definition of $q(y)$ respectively. Then $\chi(x) \in p(x)$ if and only if $\psi(y) \in q(y)$.*[22] *Symmetry*

Proof. By way of contradiction, we may suppose wlog that $\chi(x) \in p(x)$ but $\neg\psi(y) \in q(y)$. We will find $c_i, d_i \in \mathcal{M}$ such that $\models \chi(c_i)$, $\models \neg\psi(d_i)$ and $\models \varphi(c_i, d_j)$ iff $i \le j$, thereby contradicting stability.

We argue by induction. Suppose that we already have c_i, d_i for $i < n$ (the base case is left as an exercise). Since $\bar{\mathcal{M}}$ is saturated, we can find $c, d \in \bar{\mathcal{M}}$ such that $c \models p$ and $d \models q$. As $\models \neg\psi(d_i)$ for $i < n$, we have $\neg\psi(x, d_i) \in p(x)$ (since $\models \neg\psi(d_i) \Rightarrow \varphi(x, d_i) \notin p(x) \Rightarrow \neg\varphi(x, d_i) \in p(x)$). Hence $\models \bigwedge_{i<n} \neg\varphi(c, d_i) \wedge \chi(c)$, as $c \models p$ and $\chi(x) \in p(x)$. We can thus apply Tarski–Vaught to find $c_n \in \mathcal{M}$ such that $\models \bigwedge_{i<n} \neg\varphi(c_n, d_i) \wedge \chi(c_n)$. So, since the c_i for $i \le n$ satisfy $\chi(x)$, $d \models q$ and $\neg\psi(y) \in q(y)$, we have $\models \bigwedge_{i \le n} \varphi(c_i, d) \wedge \neg\psi(d)$. We again apply Tarski–Vaught to find $d_n \in \mathcal{M}$ such that $\models \bigwedge_{i \le n} \varphi(c_i, d_n) \wedge \neg\psi(d_n)$. Thus for all $i, j \le n$ we have $\models \varphi(c_i, d_j)$ iff $i \le j$ and so we are done. □

Lemma 1.8. *Let $\varphi(x, y)$ be stable and let $A \subseteq \mathcal{M}$ be algebraically closed (i.e. $\mathrm{acl}(A) = A$). Suppose that $p_1(x), p_2(x) \in S_\varphi(\mathcal{M})$ are both definable over A (i.e. the φ-definitions of p_1 and p_2 are both $\mathcal{L}_A$-formulae) and that $p_1 \upharpoonright A = p_2 \upharpoonright A$. Then $p_1 = p_2$.* *Uniqueness/ stationarity*

Proof. Let $\psi_1(y)$ and $\psi_2(y)$ be the φ-definitions of p_1 and p_2 respectively. We have $\psi_1(y), \psi_2(y) \in \mathcal{L}_A$ by assumption. $\psi_1(y)$ and $\psi_2(y)$ are also both φ^*-formulae (see Comment 1.B). It suffices to show $\mathcal{M} \models \forall y\, \psi_1(y) \leftrightarrow \psi_2(y)$. Suppose $\mathcal{M} \models \psi_1(b)$. We want to show $\mathcal{M} \models \psi_2(b)$. Let $q(y) := \mathrm{tp}(b/A) \in S(A)$. By Lemma 1.6 there exists $q'(y) \in S_{\varphi^*}(\mathcal{M})$ which is definable over $\mathrm{acl}(A) = A$ and consistent with $q(y)$. Let $\chi(x)$ be the φ^*-definition of $q'(y)$. As $\psi_1(y)$ is a φ^*-formula over A and $\mathcal{M} \models \psi_1(b)$, we have $\psi_1(y) \in q'(y)$. Thus $\chi(x) \in p_1$ by Lemma 1.7. But $\chi(x) \in \mathcal{L}_A$ and so $\chi(x) \in p_1 \upharpoonright A = p_2 \upharpoonright A$. Thus by Lemma 1.7 again we have $\psi_2(y) \in q'(y)$. But $\psi_2(y) \in \mathcal{L}_A$ and so $\psi_2(y) \in q(y) = \mathrm{tp}(b/A)$, i.e. $\mathcal{M} \models \psi_2(b)$, which was what we wanted. □

[20] Note that $p(x)$ is a complete *standard* type over A, **not** a complete φ-type (cf. Definition 1.3). $S(A)$ denotes the space of complete standard types over A.

[21] Isomorphisms act on everything: elements, types, words... So, since Y_0 is determined by Y and Y is setwise invariant under $\mathrm{Aut}(\mathcal{M}/A)$, Y_0 is also be setwise invariant under $\mathrm{Aut}(\mathcal{M}/A)$. Check the details if you're not convinced.

[22] Cf. Comment 1.B(ii).

Lemma 1.9. *Let $\varphi(x,y)$ be stable, $\mathcal{M} \subseteq \mathcal{N}$ be models and $p(x) \in S_\varphi(\mathcal{N})$. Then $p(x)$ is definable over $\mathcal{M}$ if and only if $p(x)$ is finitely satisfiable in $\mathcal{M}$.*[23]

Proof. We may assume that $\mathcal{N}$ is saturated.

($\Leftarrow$) Let $\psi(y)$ be the φ-definition of $p(x)$ and suppose that $p(x)$ is finitely satisfiable in $\mathcal{M}$. We want to show that $\psi(y) \in \mathcal{L}_\mathcal{M}$. By way of contradiction, suppose that $\psi(y) \notin \mathcal{L}_\mathcal{M}$. Then, since $\psi(y) \in \mathcal{L}_M$ iff $\psi(\mathcal{N})$ is $\mathrm{Aut}(\mathcal{N}/\mathcal{M})$-invariant, there exist $b, c \in \mathcal{N}$ such that $\mathrm{tp}(b/\mathcal{M}) = \mathrm{tp}(c/\mathcal{M})$ and $\models \psi(b) \wedge \neg\psi(c)$ (the details are left as an exercise). Thus $\varphi(x,b) \wedge \neg\varphi(x,c) \in p(x)$. But $\varphi(x,b) \wedge \neg\varphi(x,c)$ is not satisfied in $\mathcal{M}$: if it were satisfied by some $a \in \mathcal{M}$, then we would have $\mathrm{tp}(b/a) \neq \mathrm{tp}(c/a)$, a contradiction.

($\Rightarrow$) Now suppose that $p(x)$ is definable by $\psi(y) \in \mathcal{L}_\mathcal{M}$ but some formula in $p(x)$ is not satisfied in $\mathcal{M}$. Without loss of generality we may take this formula to be $\varphi(x,y)$. Let $\models \psi(b)$. By Tarski–Vaught we can find $b_1 \in \mathcal{M}$ such that $\models \psi(b_1)$. Let $a_1 \in \mathcal{M}$ be such that $\models \varphi(a_1, b_1)$. Note that $\models \neg\varphi(a_1,b_1) \wedge \psi(b)$. Again using Tarksi–Vaught, we can find $b_2 \in \mathcal{M}$ such that $\models \neg\varphi(a_1,b_2) \wedge \psi(b_2)$. $\varphi(x,b_1) \wedge \varphi(x,b_2) \in p(x)$, so let $a_2 \in \mathcal{M}$ be such that $\models \varphi(a_2,b_1) \wedge \varphi(a_2,b_2)$. Again we have $\models \neg\varphi(a_2, b)$. We continue in this way to find $a_1, b_1, a_2, b_2, \ldots \in \mathcal{M}$ such that $\models \varphi(a_i, b_j)$ iff $i \leq j$, contradicting stability. ☐

Note. The above proof adapts to the following: $p(x)$ is definable over $\mathcal{M}$ iff $p(x) \cup p_0(x)$ is finitely satisfiable in $\mathcal{M}$ for any $p_0 \in S(\mathcal{M})$ that is consistent with $p(x)$.

Unstable counterexample. Let $\mathcal{M} = (\mathbb{R}, <) \prec \mathcal{N}$ and let $\varphi(x,y)$ be $x < y$. Define $p(x) := \{x > b : b \in \mathcal{N}\} \in S_\varphi(\mathcal{N})$. $p(x)$ is definable over $\mathcal{N}$ by the formula $b = b$, since $x > b \in p(x)$ iff $b = b$ for all $b \in \mathcal{N}$. However, $p(x)$ is not finitely satisfiable in $\mathcal{M}$: pick $c \in \mathcal{N}$ such that $c > a$ for all $a \in \mathbb{R}$.

More generally, suppose that $\mathcal{M} \prec \mathcal{N}$ are models of DLO (or RCF) where $\mathcal{N}$ is saturated. Let $p(x) \in S_1(\mathcal{N})$ be $\mathrm{Aut}(\mathcal{N}/\mathcal{M})$-invariant. Then there is a dichotomy: either $p(x)$ is definable over $\mathcal{M}$ or $p(x)$ is finitely satisfiable in $\mathcal{M}$.

We now come to two very important concepts, dividing and forking:

Dividing and forking

Definition 1.10. (No stability is assumed.) Let $A \subseteq \bar{\mathcal{M}}$.

(i) A formula $\varphi(x,b)$ *divides over* A iff there exists an A-indiscernible sequence $(b_i)_{i<\omega}$ with $b_0 = b$ such that $\{\varphi(x,b_i) : i < \omega\}$ is inconsistent, i.e. $\models \neg\exists x\, \varphi(x,b_0) \wedge \cdots \wedge \varphi(x,b_n)$ for some $n < \omega$.[24]

Example 1. If $b \in \mathrm{acl}_{\bar{\mathcal{M}}}(A)$ and $\varphi(x,b)$ is consistent (i.e. $\models \exists x\, \varphi(x,b)$), then $\varphi(x,b)$ does not divide over A.

Proof. Since $b \in \mathrm{acl}_{\bar{\mathcal{M}}}(A)$, there exists $\theta(x) \in \mathcal{L}_A$ such that $\models \theta(b)$. Let $(b_i)_{i<\omega}$ be a sequence of A-indiscernibles with $b = b_0$. Then $\models \theta(b_i)$ for every $i < \omega$ (by A-indiscernibility). But $\theta(\bar{\mathcal{M}}) := \{a \in \bar{\mathcal{M}} : \models \theta(a)\}$ is finite by the definition of $\mathrm{acl}_{\bar{\mathcal{M}}}(A)$ and so only finitely many of the b_i can be distinct, contradicting Comment 1.E(i) below. ☐

Notice that any *in*consistent $\varphi(x,b)$ divides over any set A, since $\models \neg\exists x\, \varphi(x,b)$ and the constant sequence $(b)_{i<\omega}$ is trivially A-indiscernible.

Example 2. If $b \notin \mathrm{acl}_{\bar{\mathcal{M}}}(A)$, then $x = b$ divides over A, since we can find distinct A-indiscernibles $b = b_0, b_1, b_2, \ldots$ and $\models \neg\exists x\, x = b_i \wedge x = b_j$ for any $i \neq j$.

Example 3. Consider the (complete) theory $\mathrm{Th}(\mathbb{Q}, <)$. In this theory the formula $a < x < b$ divides over $\varnothing$ for any $a < b$.

[23] A set of formula $\Sigma(x)$ is said to be *finitely satisfiable* in $\mathcal{M}$ iff every finite subset of Σ is realised in $\mathcal{M}$. In the case of a complete φ-type $p(x)$, this is equivalent to every formula $\chi(x) \in p$ having a solution in $\mathcal{M}$, since the conjunction of a finite set of φ-formulae in p is itself a φ-formula in p.

[24] A sequence $(b_i)_{i\in I}$ is *A-indiscernible* iff for any formula $\psi(x_1, \ldots, x_n) \in \mathcal{L}_A$ and for any $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$ in I, we have $\models \psi(b_{i_1}, \ldots, b_{i_n}) \leftrightarrow \psi(b_{j_1}, \ldots, b_{j_n})$.

Proof. Th($\mathbb{Q}, <$) has quantifier elimination and so tp($ab/\varnothing$) is entirely determinded by the relation $a < b$.[25] So pick a_i, b_i such that $a < b < a_1 < b_1 < a_2 < b_2 < \cdots$, i.e. the interval (a_i, b_i) shifts along to the right with each step. Then the a_i, b_i are $\varnothing$-indiscernible and the formulae $a_i < x < b_i$ are pairwise inconsistent. □

(ii) A formula $\varphi(x, b)$ *forks over* A iff $\varphi(x, b)$ implies (w.r.t. the ambient theory T) a finite disjunction $\bigvee_{i=0}^{k} \psi_i(x, c_i)$ such that each $\psi_i(x, c_i)$ divides over A.

(iii) Let $\Sigma(x, \bar{b})$ be a partial type that is closed under finite conjunctions.[26] $\Sigma(x, \bar{b})$ *divides/forks over* A iff some formula in $\Sigma(x, \bar{b})$ divides/forks over A.

Comment 1.E.

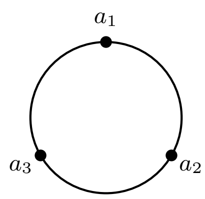
(i) Suppose that a consistent formula $\varphi(x, b)$ divides over A. Let the sequence $(b_i)_{i<\omega}$ witness the dividing. Then the b_i are pairwise distinct.

Proof. Suppose there exist $i < j$ such that $b_i = b_j$. Then by A-indiscernibility we must have $b_{i'} = b_{j'}$ for any $i' < j'$. Thus the sequence $(b_i)_{i<\omega}$ is constant and so $\{\varphi(x, b_i) : i < \omega\} = \{\varphi(x, b)\}$, which is consistent, a contradiction. □

(ii) $\Sigma(x, \bar{b})$ divides over A if there exist A-indiscernibles $\bar{b} = \bar{b}_0, \bar{b}_1, \bar{b}_2, \ldots$ such that $\bigcup_{i<\omega} \Sigma(x, \bar{b}_i)$ is inconsistent.

(iii) In general, dividing ⇒ forking (since a formula implies itself). We will see that if T is stable,[27] then in fact forking ⇒ dividing. The following example shows that this does not hold in general, however:

Example. Consider the language $\mathcal{L} = \{B\}$, where B is a ternary relation symbol. Let $\mathcal{C}$ be a circle considered as an $\mathcal{L}$-structure, where B is interpreted as clockwise betweenness on the circle; that is, $\mathcal{C} \models B(a, b, c)$ iff b lies on the clockwise arc from a to c and a, b, c are distinct. We work in Th($\mathcal{C}$) (a complete theory). Let $a_1, a_2, a_3 \in \mathcal{C}$ split the circle up into three arcs, for example: Forking $\not\Rightarrow$ dividing



Now, the formula $x = x$ does not divide over $\varnothing$. It is also stable. However, $x = x$ implies $\bigvee_{i \neq j} B(a_i, x, a_j)$, since all points on $\mathcal{C}$ lie between two of a_1, a_2, a_3 (including a_1, a_2, a_3 themselves), and each $B(a_i, x, a_j)$ divides over $\varnothing$ by very similar reasoning to that in Example 3 above. So $x = x$ is stable, does not divide over $\varnothing$, but does fork over $\varnothing$.

Proposition 1.11. *Let $\varphi(x, y)$ be stable, $A \subseteq \mathcal{M}$ and $p(x) \in S_\varphi(\mathcal{M})$. Then the following are equivalent:*

(i) $p(x)$ is definable over acl(A).

(ii) $p(x)$ does not divide over A.

[25] For arbitrary tuples a, b, c and a set A, tp(ab/Ac) is shorthand for tp($(a, b)/A \cup \{c\}$).
[26] Recall that $\bar{b}$ denotes a (possibly) infinite tuple.
[27] A complete theory T is *stable* iff all formulae $\varphi(x, y)$ are stable in T (no matter how the free variables of φ are partitioned).

Proof. (i) ⇒ (ii) By passing to $\bar{\mathcal{M}}$ we may assume that $\mathcal{M}$ is saturated. As $p(x)$ is definable over $\mathrm{acl}(A)$, $p(x)$ is $\mathrm{Aut}(\mathcal{M}/\mathrm{acl}(A))$-invariant. Consider some $\psi(x,c) \in p(x)$. Since $\mathcal{M}$ is saturated, we can find an A-indiscernible sequence $c = c_0, c_1, c_2, \dots$ in $\mathcal{M}$. Then (exercise) $\mathrm{tp}(c_i/\mathrm{acl}(A))$ is the same for all $i < \omega$. By the $\mathrm{Aut}(\mathcal{M}/\mathrm{acl}(A))$-invariance of $p(x)$, we have $\psi(x,c_i) \in p(x)$ for all $i < \omega$ and thus $\{\psi(x,c_i) : i < \omega\} = \{\psi(x,c)\}$. So $\psi(x,c)$ does not divide over A and hence $p(x)$ also does not divide over A.

¬(i) ⇒ ¬(ii) We can again assume $\mathcal{M}$ to be saturated. Assume that $p(x)$ is not definable over $\mathrm{acl}(A)$. Let $p_0 = p(x) \restriction \mathrm{acl}(A)$. By Lemmas 1.6 and 1.8 there exists a unique extension $p_1(x) \in S_\varphi(\mathcal{M})$ of $p_0(x)$ that is definable over $\mathrm{acl}(A)$. So $p(x) \neq p_i(x)$. Without loss of generality we have some $\varphi(x,b) \in p(x)$ but $\neg\varphi(x,b) \in p_1(x)$ (the proof for the other case is symmetric). We will show that $p_0(x) \cup \{\varphi(x,b)\}$ divides over A, thereby showing that $p(x)$ divides over A. Let $q_0(y) = \mathrm{tp}(b/\mathrm{acl}(A))$. By Lemma 1.6 we can find $q(y) \in S_{\varphi^*}(\mathcal{M})$ that is definable over $\mathrm{acl}(A)$ and consistent with $q_0(y)$. Now take a small model $\mathcal{M}_0 \prec \mathcal{M}$ such that $A \subseteq \mathcal{M}_0$ (or $\mathrm{acl}(A) \subseteq \mathcal{M}_0$, it doesn't matter). So $q(y)$ is definable over $\mathcal{M}_0$. By the note after Lemma 1.9, $q(y) \cup q_0(y)$ is finitely satisfiable in $\mathcal{M}_0$. Hence (exercise) $q(y) \cup q_0(y)$ extends to a complete type $q^*(y) \in S(\mathcal{M})$ that is also finitely satisfiable in $\mathcal{M}_0$. Let $\chi(x)$ be the φ^*-definition of $q(y) \in S_{\varphi^*}(\mathcal{M})$. By the proof of Lemma 1.2, in particular the moreover clause, there are $b_0, b_1, b_2, \dots$ such that $b_0 \models q^* \restriction \mathcal{M}_0$ and $b_{i+1} \models q^* \restriction M, b_0, \dots, b_i$ for all $i < \omega$ and such that $\chi(x)$ is a positive Boolean combination of the $\varphi(x,b_i)$'s, i.e. the $\varphi^*(b_i,x)$'s. As q^* is finitely satisfiable in $\mathcal{M}_0$, $(b_i)_{i<\omega}$ is indiscernible over $\mathcal{M}_0$ and hence also over A.

Interlude. New notation, new p's and q's.[28] Suppose that $\mathcal{M}$ is saturated and that $p(x) \in S(\mathcal{M})$ is $\mathrm{Aut}(\mathcal{M}/\mathcal{M}_0)$-invariant for some small $\mathcal{M}_0 \prec \mathcal{M}$, i.e. whether or not $\varphi(x,b) \in p(x)$ depends on $\mathrm{tp}(b/M_0)$, e.g. $p(x)$ is definable over $\mathcal{M}_0$ because $p(x)$ is finitely satisfiable in $\mathcal{M}_0$. In $\mathcal{M}$, let $a_0 \models p \restriction \mathcal{M}_0$ and $a_{i+1} \models \mathcal{M}_0, a_0, \dots, a_i$ for all $i < \omega$. Then $(a_i)_{i<\omega}$ is indiscernible over $\mathcal{M}_0$ and $\mathrm{tp}(a_0, a_1, \dots/\mathcal{M}_0)$ depends on p and $\mathcal{M}_0$.

Also note that if $p_0(x) \in S(\mathcal{M}_0)$, then by compactness $p_0(x)$ has an exntension to $p(x) \in S(\mathcal{M})$ which is finitely satisfiable in $\mathcal{M}_0$. We apply the above construction to get an $\mathcal{M}_0$-indiscernible sequence $(a_i)_{i<\omega}$. End of interlude.

Now, $\{\neg\chi(x)\} \cup \{\varphi(x,b_i) : i < \omega\}$ is inconsistent. Let $\psi(y)$ be the φ-definition of $p_1(x)$. $\mathrm{tp}(b/\mathrm{acl}(\mathcal{M})) \cup q(y)$ is consistent, $\models \neg\psi(b)$ and $\psi(y)$ is a φ^*-formula, so $q(y) \notin q(y)$. By Lemma 1.7 we habe that $\chi(x) \notin p_1(x)$. But $\chi(x)$ is over $\mathrm{acl}(A)$ and is a φ-formula and so $\neg \in p_0(x) = p_1 \restriction \mathrm{acl}(A)$. So we've shown that $p(x) \cup \{\varphi(x,b_i) : i < \omega\}$ is inconsistent and that $(b_i)_{i<\omega}$ is indiscernible over $\mathcal{M}_0$ and thus also over A. So, to see that $p_0(x) \cup \{\varphi(x,b)\}$ divides over A, it suffices to show that $\mathrm{tp}(b/\mathrm{acl}(A)) = \mathrm{tp}(b_i/\mathrm{acl}(A))$. But we have this, since $q_0 = \mathrm{tp}(b/\mathrm{acl}(A))$, $q_0(y) \subseteq q^*(y)$ and $b_i \models q^* \restriction \mathcal{M}_0$ and hence $b_i \models q^* \restriction \mathrm{acl}(A)$. □

Remark 1.12. *Let $\varphi(x,y)$ be stable, $A \subseteq \mathcal{M}$ and $b \in \mathcal{M}$. Then the following are equivalent:*

(i) There exists $p(x) \in S_\varphi(\mathcal{M})$ which is definable over $\mathrm{acl}(A)$ and contains $\varphi(x,b)$.

(ii) $\varphi(x,b)$ does not divide over A.

Proof. (i) ⇒ (ii) Use the (i) ⇒ (ii) part of Proposition 1.11.

(ii) ⇒ (i) Adapt the (i) ⇒ (ii) part of Proposition 1.11. Let $q_0(y) = \mathrm{tp}(b/\mathrm{acl}(A))$. Consider an extension $q(y) \in S_\varphi(\mathcal{M})$ of $q_0(y)$ whch is definable over $\mathrm{acl}(A)$. Let $\chi(x)$ be the φ^*-definition of $q(y)$. Show that $\chi(x)$ is inconsistent. □

Δ-types **Comment 1.F.** Everything we have done regarding φ-types for stable $vphi(x,y)$ extends to Δ-types, where Δ is a finite set of stable formulae $\varphi(x,y_0), \dots, \varphi(x,y_k)$.

Proof. This can be seen directly or through so-called *coding through case analysis*: Add constants $1, \dots, k$ (just two would in fact suffice, we could take different lengths of constants). Define $\psi(x, y_1, \dots, y_k, z)$ to be the following formula:

$$(z = 0 \vee \dots \vee z = k) \wedge (z = i \rightarrow \varphi_i(x, y_i)) \tag{1.2}$$

[28] In stability theory, as in life, one should always mind one's p's and q's.

So, for example, $p(x) \in S_\Delta(\mathcal{M})$ is definable over $\mathrm{acl}(A)$ iff for all i, $p(x) \restriction \varphi_i$ is definable over $\mathrm{acl}(A)$. □

Corollary 1.13. *Suppose that $\varphi_1(x,y)$ and $\varphi_2(x,z)$ are both stable formulae that each divide over A. Then $\varphi_1(x,y) \vee \varphi_2(x,y)$ divides over A.*

Proof. Note that $\psi_i(x,y) \vee \psi_2(x,z)$ is stable (Comment 1.A(ii)). Let $\Delta = \{\psi_1, \psi_2\}$. Suppose that $\varphi_1(x,y) \vee \varphi_2(x,y)$ does not divide over A. By Remark 1.12, there exists $p(x) \in S_\Delta(\bar{\mathcal{M}})$ containing $\varphi_1(x,b) \vee \varphi_2(x,c)$ which is definable over $\mathrm{acl}(A)$. Either $\psi_1(x,b) \in p(x)$ or $\psi_2(x,c) \in p(x)$ (or both). Since $p(x)$ is $\mathrm{Aut}(\bar{\mathcal{M}}/\mathrm{acl}(A))$-invariant, in the first case $\psi_1(x,b)$ does not divide over A, while in the second case $\psi_2(x,c)$ does not divide over A (again by Remark 1.12). In both cases we have a contradiction. □

Lemma 1.6*. *Let Δ be a finite set of stable formulae $\varphi_i(x,y_i)$, $A \subseteq \mathcal{M}$ and $p(x) \in S(A)$. Then there exists $q(x) \in S_\Delta(\mathcal{M})$ such that $p(x) \cup q(x)$ is consistent and $q(x)$ is definable over* $\mathrm{acl}(A)$, *i.e. for each i, $q(x) \restriction \varphi_i$ is definable over* $\mathrm{acl}(A)$.

Proof. This follows from Lemma 1.6 and Comment 1.F. □

Proposition 1.14. *Suppose that T is stable (see footnote 27), $p(x) \in S(A)$ and $A \subseteq \mathcal{M}$. Then there exists an* $\mathrm{acl}(A)$*-definable extension $p'(x) \in S(\mathcal{M})$ of $p(x)$.*

Proof. By passing to $\bar{\mathcal{M}}^{\mathrm{eq}}$, we may assume that A is algberaically closed. By Lemma 1.6, for each $\varphi(x,y)$ there exists $p(x)^\varphi \in S_\varphi(\mathcal{M})$ definable over $\mathrm{acl}(A)$ and consistent with $p(x)$, in particular $p(x) \restriction A \subseteq p(x)$. Note that by Lemma 1.8, $p(x)^\varphi \restriction A$ has a unique extension $q(x) \in S_\varphi(\mathcal{M})$ definable over $\mathrm{acl}(A)$. Lemma 1.6* tells us that if $\Delta = \{\varphi_1(x,y_1), \ldots, \varphi_n(x,y_n)\}$, then there exists a $p(x)^\Delta \in S_\Delta(\mathcal{M})$ definable over A and consistent with $p(x)$. So by the uniqueness just mentioned, $p(x)^\Delta \restriction \varphi_i = p(x)^{\varphi_i}$ for all $i = 1, \ldots, n$. Hence by compactness $\bigcup_{\varphi(x,y)} p(x)^\varphi$ is a complete type $q(x) \in S(\mathcal{M})$, $p(x) \subseteq q(x)$ and $q(x)$ is definable over A (the details are left as an exercise). □

Proposition 1.15. *Let T be stable, $p(x) \in S(B)$ and $A \subseteq B$. Then the following are equivalent:*

(i) $\varphi(x,b)$ does not divide over A.

(ii) $\varphi(x,b)$ does not fork over A.

(iii) There exists $\mathcal{M} \supseteq B$ and an $\mathrm{acl}(A)$*-definable extension $p'(x) \in S(\mathcal{M})$ of $p(x)$.*

If $p(x) = \mathrm{tp}(c/B)$, then (1)–(3) say that *c is independent from B over A*, which we denote $c \mathop{\smile\!\!\!\!\!\!\mid}_A B$.

Proof. (i) ⇒ (ii) Suppose that $\varphi(x,y) \in p(x)$ forks over A. So $\varphi(x,b) \to \psi_1(x,c_1) \vee \psi_2(x,c_2)$, where each $\psi_u(x,c_i)$ divides over A. But $\psi_1(x,y_1)$ and $\psi_2(x,y_2)$ are both stable and so by Corollary 1.13 $\psi_1(x,c_1) \vee \psi_2(x,c_2)$ divides over A. Thus $\varphi(x,b)$ divides over A, a contradiction.

(ii) ⇒ (iii) (This implication is the reason for introducing the notion of forking.) If $p(x) \in S(B)$ does not fork over A, then $p(x) \cup \{\neg\psi(x,c) : c \in \mathcal{M}, \psi(x,c) \text{ divides over } A\}$ is consistent and thus exntends to $p'(x) \in S(\mathcal{M})$ which does not divide over A. By Proposition 1.11, $p'(x)$ is definable over $\mathrm{acl}(A)$.

(iii) ⇒ (i) This follows from the (i) ⇒ (i) of Proposition 1.11. □

Conclusion 1.16. *Let T be stable. Then:*

(i) Finite character: *$p(x) \in S(B)$ does not fork over $A \subseteq B$ if $p(x) \restriction A\bar{b}$ does not fork over A for all finite $\bar{b} \in B$.*

(ii) Transitivity: *Let $A \subseteq B \subseteq C$ abd $p(x) \in S(C)$. If $p(x)$ does not fork over B and $p(x) \restriction B$ does not fork over A, then $p(x)$ does not fork over A.*

(iii) Symmetry: *$a \mathop{\smile\!\!\!\!\!\!\mid}_A b$ iff $b \mathop{\smile\!\!\!\!\!\!\mid}_A a$.*

(iv) Any $p(x) \in S(A)$ has a global non-forking extension $p'(x) \in S(\bar{\mathcal{M}})$.

(v) If A is algebraically closed, then the global non-forking extension $p'(x) \in S(\bar{\mathcal{M}})$ is unique.

(vi) Let $p(x) \in S(A)$*. Then set of global non-forking extensions in* $S(\bar{\mathcal{M}})$ *of* $p(x)$ *are conjugates over* $\mathrm{Aut}(\bar{\mathcal{M}}/A)$.

Proof. Exercise: use various previous results and/or consult [5] and [1]. □

Definition 1.17. A theory T is *simple* iff for every $p(x) \in S(\bar{\mathcal{M}})$, $p(x)$ does not divide over small $A \subseteq \mathcal{M}$, $|A| \leq |T|$.

Stable implies simple, but the converse does not hold. (1)–(4) in Conclusion 1.16 hold for simple T, while (1)–(4) + (5) characterise stability.

Chapter 2

Stable Group Theory

Goal: Give an account of the most general form of stable group theory, i.e. type definable homogeneous spaces. Our work will be in line with [5], except with corrections (in particular, a correction of [5, Lemma 1.6.4]).

Stable group theory $\sim$ equivariant stability ('there's a group action around') $\sim$ forking in the presence of a group action. This is a part of pure model theory.

Definition (G, X) is a homogeneous space if (G, X) is a transitive (has only one orbit) group action. *Homogeneous space*

Fix $x_0 \in X$ and let $H = \mathrm{Stab}(x_0)$. Then there is a G-invariant bijection between G/H (the set of left cosets of H) and X. Explicitly, $gH \in G/H \mapsto gx_0$, and in the other direction, $x \in X \mapsto gH$ where g satisfies $gx_0 = x$; such a g exists by transitivity. One routinely checks that this is well-defined and indeed G-invariant.

In different categories, homogeneous spaces give different objects of interest:

1. Topological spaces: G is a topological group, X a topological space, action is continuous. (In this context, it is more interesting to deal with actions with dense orbits, instead of actions with one orbit). Here, H is a closed subgroup.
2. Differentiable manifolds: G is a Lie group, X a differentiable manifold, C^∞ action.
3. Algebraic varities: G algebraic group, X algebraic variety, action is 'regular', H algebraic subspace, G/H is a smooth algebraic variety.
4. Model theory, i.e. category of definable over A sets: This is what we will be studying!

Definition 2.1. Let T be a complete theory, $\bar{M}$ a saturated model, and A, B be "small" sets of parameters. *Type-definability/ ∞-definability*

(i) $X \subseteq \bar{M}^n$ (or $\in \bar{M}^{\mathrm{eq}}$) is type-definable, or $\bigwedge$-definable, or ∞-definable, over A if X is the set of solutions of a partial type $\Sigma(\bar{x}) \in L_A$.

(ii) A homogeneous space (G, S) is type definable over A if the set G, the set S, and the graphs of the group operations and of the action are all type definable over A.

* * *
Romantic Interlude (but no forking... :P) There's an analogy between automorphism groups and definable groups. Let M be a saturated structure, G be definable in M, and (G, X) be a regular action (1-orbit and $gx = hx \Rightarrow g = h$). Let M' be a 2-sorted structure (M, X) in which the action is definable. Then, $\mathrm{Aut}(M'/M)$ is plainly (isomorphic to) a subgroup of the permutations of X. Furthermore, if $\sigma \in \mathrm{Aut}(M'/M)$ and $g \cdot x = y$, then $g \cdot \sigma(x) = \sigma(y)$, i.e. σ must commute with the G-action. The converse also holds, so we have that $\mathrm{Aut}(M'/M)$ is isomorphic to G^{op}, the set of permutations of X which commute with the G-action.

Via this isomorphism, it can be seen that various notions in each area are actually analogues, for example, strong types and G^0, or, Lascar strong types and G^{00}.
* * *

Remark When T is stable, a ∞-definable group is an intersection of definable groups. In general, this is not true. We provide an example of a ∞-definable group which is not an intersection of a *nested sequence* of definable groups. Take a saturated model of $(\mathbb{R}, <, +\restriction[-1,1], q \in \mathbb{Q})$. Then $\{a \in M : -\frac{1}{n} < a < \frac{1}{n}, n = 1, 2, 3, \ldots\}$ is the example.

Question Is there a theory with a ∞-definable infinite group but no definable infinite groups?

Remark 2.2. *Let (G, S) be ∞-definable over A. Then the group operation is given by the restriction of some A-definable partial function to $G \times G$. Similarly for the action.*

Proof. Let $\Sigma(x, y, z)$ be the partial type defining the group operation. The goal is to find a formula $\sigma(x, y, z)$ which is the graph of a partial function on $\bar{M}$ such that $\sigma(\bar{M}^3) \cap G^3 = \Sigma(\bar{M}^3)$. This is easy to achieve via compactness; the details are left as an exercise for the reader. □

Relative definability

Definition Let (G, S) be a ∞-definable over A homogeneous space. $X \subseteq S$ is relatively definable if it is of the form $\{a \in S : \psi(a)\}$, where $\psi(x)$ is a formula with parameters.

Remark: The relatively definable subsets of S form a Boolean algebra.

Remark: If X is a relatively definable subset of $S \times S$, then it is not necessarily the case that $\pi(X) \subseteq S$ is relatively definable, even if stability is assumed.

Generic subsets and generic types

Definition 2.3.

(i) A relatively definable subset $X \subseteq S$ is generic if there exists $g_1, ..., g_n$ s.t. $g_1 X \cup \ldots \cup g_n X = S$.

(ii) A global type $p(x) \in S_S(\bar{M})$ (note that $p(x) \models x \in S$) is generic if every formula $\psi(x) \in p(x)$ defines a generic subset, i.e. $\psi(\bar{M}) \cap S$ is generic.

Our first aim is to prove the existence of generic types in a stable theory. Before we can do this, we need a couple more results from local stability theory.

Lemma 2.4. *Let $\varphi(x, y)$ be a stable formula, A be a set of parameters, b an element. Then TFAE:*

(i) $\varphi(x, b)$ does not divide over A.

(ii) Some positive boolean combination of A-conjugates of $\varphi(x, b)$ is consistent and definable over A (i.e. equivalent to a formula with parameters in A).

Remark: An A-conjugate of $\varphi(x, b)$ is a formula $\varphi(x, b')$ where $\mathrm{tp}(b'/A) = \mathrm{tp}(b/A)$ (or where there is a $\sigma \in \mathrm{Aut}(\bar{M}/A)$ s.t. $\sigma(b) = b'$.)

Remark: $\varphi(x)$ is 'over A' or equivalent to a formula with parameters in A iff $\varphi(\bar{M})$ is $\mathrm{Aut}(\bar{M}/A)$ invariant.

Motto of Lemma: 'A formula that doesn't divide isn't very far from a formula definable over A.'

Proof. *(ii)* ⇒ *(i)*. Let $\psi(x)$ over A be given as in (ii), so $\psi(x)$ is a consistent φ-formula over A. Hence, let $p(x) \in S_\varphi(A)$ contain $\psi(x)$. By Lemma 1.6, let $p^*(x) \in S_\varphi(\bar{M})$ extend p and be definable over $\mathrm{acl}(A)$.

Now, $\psi(x)$ is a finite disjunction of finite conjunctions of translates $\varphi(x, b')$ of $\varphi(x, b)$, and, $\psi(x) \in p^*$. Hence, $\varphi(x, b') \in p^*(x)$ for some b' s.t. $\mathrm{tp}(b'/A) = \mathrm{tp}(b/A)$. By (easy part of) Proposition 1.11, $\varphi(x, b')$ does not divide over A, and so $\varphi(x, b)$ does not divide over A.

(i) ⇒ *(ii)*. Assume (i) holds. Let $p(x) \in S_\varphi(\bar{M})$ contain $\varphi(x, b)$ and be definable over $\mathrm{acl}(A)$. Let $q(y) \in S_{\varphi^*}(\bar{M})$ be definable over A and consistent with $\mathrm{tp}(b/\mathrm{acl}(A)) = q_0(y)$. Let $\psi(y)$ be φ-definition of $p(x)$ and $\sigma(x)$ be φ^*-definition of $q(y)$.

Since $\varphi(x, b) \in p(x)$, $\psi(b)$ holds. Since q is consistent with $q_0(y)$, we then get that $\psi(y) \in q$. Then by Lemma 1.7, $\sigma(x) \in p(x)$. In particular, $\sigma(x)$ is consistent, and we know it is over $\mathrm{acl}(A)$. By (moreover clause of) Lemma 1.2, $\sigma(x)$ is a positive Boolean combination of formulas $\varphi(x, b')$ with $b' \in \bar{M}$ realising $q_0(y)$, i.e. of $\mathrm{acl}(A)$-conjugates, and hence of A-conjugates, of $\varphi(x, b)$.

$\sigma(x)$ is definable over $\mathrm{acl}(A)$, but not necessarily over A. This is resolved by taking the disjunction of all A-conjugates of $\sigma(x)$ (which will be a finite disjuction by the finiteness in the definition of algebraic closure), which will be definable over A. □

Lemma 2.5. *Let T be any theory and let $\varphi(x,y)$ be stable. Suppose that $\varphi(x,b)$ divides over A. Let $q(y) \in \mathrm{tp}_{\varphi^*}(b/A)$, $q^*(y) \in S_{\varphi^*}(\bar{M})$ an extension of $q(y)$ definable over* $\mathrm{acl}(A)$. *Then there is a k s.t. whenever $(b_i : i < \omega)$ are s.t. $b_0 \models q^*(y)\restriction \mathrm{acl}(A), \ldots, b_{n+1} \models q^*(y)\restriction \mathrm{acl}(A, b_0, ..., b_n)$, then $\{\varphi(x, b_i) : i < \omega\}$ is k-inconsistent.*

Proof. First show that $\{\varphi(x, b_i) : i < \omega\}$ is inconsistent. Suppose not, so for some b_i as in the statement of the lemma, $\{\varphi(x, b_i) : i < \omega\}$ is consistent. Let $p(x) \in S_\varphi(A)$ s.t. $p(x) \cup \{\varphi(x, b_i) : i < \omega\}$ is consistent. Let $p^*(x) \in S_\varphi(\bar{M})$ s.t. $p^*(x)$ extends $p(x)$ and is definable over $\mathrm{acl}(A)$.

Let $r(x) \in S_\varphi(A, b_i)_{i<\omega}$ extend $p(x) \cup \{\varphi(x, b_i) : i < \omega\}$. Let $r^*(x) \in S_\varphi(\bar{M})$ extend $r(x)$ and be definable over $\mathrm{acl}(A, b_i)_{i<\omega}$.

Let $\chi(x)$ be the φ^* definition of $q^*(y)$, so $\chi(x)$ is a φ-formula over $\mathrm{acl}(A)$.
Let $\theta(y)$ be the φ definition of $p^*(x)$, so $\theta(y)$ is a φ^*-formula over $\mathrm{acl}(A)$.
Let $\psi(y)$ be the φ definition of $r^*(x)$, so $\psi(y)$ is a φ^*-formula over $B_0 = \mathrm{acl}(A, b_0, ..., b_n)$, some n.

Now, $\neg\varphi(x, b) \in p^*(x)$ by Remark 1.12, so $\neg\theta(b)$, so $\neg\theta(y) \in q^*(y)$, so by symmetry $\neg\chi(x) \in p^*(x)$, so $\neg\chi(x) \in p(x)$, so $\neg\chi(x) \in r(x) \subset r^*(x)$, so by symmetry $\neg\psi(y) \in q^*(y)$. Now, $\neg\psi(y)$ is over B_0 and $b_n \models q^*\restriction B_0$, so $\models \neg\psi(b_n)$, so $\neg\varphi(x, b_n) \in r^*$, contradiction.

Now we want to find the finite bound k. It evidently looks like a compactness argument will be needed. However, since '$x \in \mathrm{acl}(A)$' is not expressible in a first-order way (as it requires an infinite disjunction over all formulas over A), we cannot use compactness immediately. So, let $A \subseteq M_0 \subseteq M_1 \subseteq \ldots$ be a sequence where M_0 is saturated over A and for every $n < \omega$, M_{n+1} is saturated over M_n.

Claim. Whenever $(b_i : i < n)$ is such that $b_0 \models q^*\restriction A, b_{i+1} \models q^*\restriction \mathrm{acl}(A, b_0, \ldots, b_i)$, there exist $b'_0, b'_1 \ldots, b'_n \equiv_A b_0, \ldots, b_n$ s.t. $b'_0 \in M_0$ realises $q^*\restriction A$ and $b'_{i+1} \in M_{i+1}$ realises $q^*\restriction M_i$.

Proof of Claim. Let b'_0 realise $\mathrm{tp}(b_0/A)$ in M_0. WMA $b'_0 = b_0$, so $b_0 \in M_0$. By Lemmas 1.6 and 1.8, $\mathrm{tp}_{\varphi^*}(b_1/\mathrm{acl}(A, b_0))$ has a unique extension to $q^{**}(y) \in S_{\varphi^*}(\bar{M})$ definable over $\mathrm{acl}(Ab_0)$ which is consistent with $\mathrm{tp}(b_1, \mathrm{acl}(Ab_0))$. But $q^*(y)$ is already such a type, so $q^{**}(y) = q^*(y)$. In particular, $q^*(y) \cup \mathrm{tp}(b_1/\mathrm{acl}(Ab_0))$ is consistent. So $q^*(y)\restriction M_0 \cup \mathrm{tp}(b_1/\mathrm{acl}(Ab_0))$ is consistent, so realised in M_1, by b'_1 say, so $\mathrm{tp}(b_0 b_1/A) = \mathrm{tp}(b'_0 b'_1/A)$. Continuing in this fashion proves the claim. □

So now suppose for contradiction that for each k, we can find $b_0, \ldots, b_k$ such that $b_0 \models q^*\restriction \mathrm{acl}(A), b_i \models q^*\restriction \mathrm{acl}(Ab_0 \ldots b_{i-1})$ for $i > 0$, and, $\{\varphi(x, b_i) : i \le k\}$ is consistent. By the claim, we can further assume that $b_i \in M_i$ for all $i \le k$, $b_0 \models q^*\restriction \mathrm{acl}(A)$, and, $b_i \models q^*\restriction M_{i-1}$ for $i > 0$.

Now we use a compactness argument: To the language add constants $c_0, c_1, c_2, \ldots$ (corresponding to the b_i), predicates $P_0, P_1, P_2, \ldots$ (corresponding to the M_i) and a constant d (corresponding to a witness of consistency of the $\varphi(x, b_i)$). To the theory add sentences to say that $A \subseteq P_0, P_0 \models \mathrm{Th}(\bar{M}, a)_{a\in A}, P_0 \prec P_1 \prec P_2 \prec \ldots$, $c_i \in P_i$ for each $i < \omega$, $c_{i+1} \models q^*\restriction P_i$ and $\varphi(d, c_i)$ for every i. This theory has a model by compactness, but this contradicts first part of this proof (that $\{\varphi(x, b_i) : i < \omega\}$ is inconsistent). □

We are now in a position to prove:

Proposition 2.6. *Assume T stable, and (G, S) a ∞-definable over A homogeneous space. Then*

(i) The collection of non-generic relatively definable subsets of S forms a proper ideal in the Boolean algebra of relatively definable subsets of S.

(ii) Let $X \subseteq S$ be relatively definable over B. Then X is non-generic iff there exists an indescernible over B sequence $(g_i : i < \omega)$ of elements $g_i \in G$ s.t. $\{g_i X : i < \omega\}$ is inconsistent (i.e. $\bigcap_{i<\omega} g_i X = \varnothing$, i.e. $\bigcap_{i\le n} g_i X = \varnothing$ for some n).

Corollary (of (i)). Assume T stable and (G, S) a ∞-definable over A homogeneous space. Then there is a global generic type.

Proof of Corollary. By (i), the set of non-generic relatively definable subsets of S forms an ideal. Hence, the set of complements of non-generic relatively definable subsets of S forms an ultrafilter. Noting that the complement of a relatively definable set is relatively definable, let $p(x)$ be the set of formulas which define the sets in this ultrafilter. Since an ultrafilter is closed under finite intersection and does not contain the empty set, $p(x)$ is finitely realisable in S. Since an ultrafilter is maximal, $p(x)$ is also maximal. Together these two facts imply that $p(x)$ is a complete type over S.

It remains to show that $p(x)$ is generic, which reduces to showing that the complement of a non-generic is generic. This follows from 'ideal-ness': if both X and its complement were non-generic, then their union is non-generic (by definition of ideal), but their union is S, which we know is generic. □

Proof of Proposition 2.6. (i) By adding constants to L, we can assume $A = \varnothing$. Trivially, S is generic and any (relatively definable) subset of a non-generic is non-generic, so we only need to show that if X, Y are both non-generic, then $X \cup Y$ is non-generic. The strategy is to find an auxiliary structure M_0 with relations R_X, R_Y and $R_{X\cup Y}$ such that: they are all stable, R_X (resp. $R_Y, R_{X\cup Y}$) divides if and only if X (resp. $R_Y, R_{X\cup Y}$) is non-generic, and, $R_{X\cup Y} \leftrightarrow R_X \vee R_Y$. This gives us the result: X and Y are non-generic implies R_X and R_Y divide, which implies $R_X \vee R_Y$ divides (by Corollary 1.13), so $R_{X\cup Y}$ divides, so $X \cup Y$ is non-generic.

Before defining M_0, note that we will only have one predicate R instead of the purported three. This is for notational convenience, since the arguments with one predicate go through when we add more predicates. So, let X be a fixed relatively definable subset of S and let M_0 be the 2-sorted structure (S, G, R) where $R \subseteq S \times G$ is defined as $R(x, y) := x \in yX$. Let $T_0 = \mathrm{Th}(M_0)$.

Claim. $R(x, y)$ is stable in T_0.

Proof of Claim. Suppose not, so we can find a_i, g_i for $i < \omega$ such that $R(a_i, g_j)$ iff $i \leq j$. By definition of M_0, this means that in our original structure $\bar{M}$ we have $a_i \in g_jX$ iff $i \leq j$. By Ramsey and compactness, WMA that $((a_i, g_i) : i < \omega)$ is an indiscernible sequence (in $\bar{M}$).

Since S is ∞-definable and X is relatively definable, X is ∞-definable, by $\{\chi_i(x) : i \in I\}$ say. Then, fixing $j < \omega$, g_jX is defined by $\{\chi_i(g_j^{-1}x) : i \in I\}$. Now fix $i < \omega$ greater than j, so that $a_i \notin g_jX$. Hence, for some fixed $i^* \in I$, we have $\neg\chi_{i^*}(g_j^{-1}a_i)$. By indiscernibility this means that for all $j < i < \omega$, we have $\neg\chi_{i^*}(g_j^{-1}a_i)$. But then we have $\chi_{i^*}(g_j^{-1}a_i)$ iff $i \leq j$, contradicting T stable. □

Note that in T_0, there is a unique 1-type over $\varnothing$ realised in the S sort (in any model), equivalently, "$x \in S$" isolates a complete type over $\varnothing$ in T_0, equivalently, the automorphism group of M_0 is transitive on S. The last equivalence is easy to prove: for every $g \in G$ we get the automorphism of M_0 $x \in S \mapsto gx$ and $h \in G \mapsto g.h$ (automorphism since $x \in yX \Rightarrow gx \in gyX$). Since G is transitive on S, we conclude that the $\mathrm{Aut}(M_0)$ is transitive on S.

Claim. X is generic in T_0 iff $R(x, 1(= \mathrm{id}_G))$ does not divide over $\varnothing$ in T_0.

Proof of Claim. $\Rightarrow$: Suppose X generic so $g_1X \cup \ldots \cup g_nX = S$ for some g_i. Therefore the formula $R(x, g_1) \vee \ldots \vee R(x, g_n) \leftrightarrow x \in S$ is in T_0. Since the g_i's have the same 1-type as 1, the $R(x, g_i)$'s are $\varnothing$-conjugates of $R(x, 1)$. So by Lemma 2.4, we get that $R(x, 1)$ does not divide over $\varnothing$.

$\Leftarrow$: Conversely, suppose $R(x, 1)$ does not divide over $\varnothing$. Then by Lemma 2.4, and working in a saturated extension M_0^*, some positive Boolean combination of $\varnothing$-conjugates of $R(x, 1)$ is consistent and over $\varnothing$. So then we get, for some g_i^*, $R(x, g_1^*) \vee \ldots \vee R(x, g_n^*) \leftrightarrow x \in S^*$, since S^* is the only non-empty definable subset of S^*. Since $M_0 \prec M_0^*$, we get the same thing in M_0, i.e. for some g_i, $g_1X \cup \ldots \cup g_nX = S$. □

But now we are done: Given X, Y relatively definable subsets of S, consider the 2-sorted structure $(S, G, R_X, R_Y, R_{X\cup Y})$ where $R_X(x, y) := x \in y.X$, $R_Y(x, y) := x \in y.Y$, $R_{X\cup Y}(x, y) := x \in y.(X \cup Y)$. The claims (and importantly, their proofs) hold for the predicates in this structure, and, we also have $R_X(x, 1) \vee R_Y(x, 1) \leftrightarrow R_{X\cup Y}(x, 1)$. Then following the argument given at the start of the proof, we conclude that if X, Y are non-generic then $X \cup Y$ must be non-generic too, thus completing the proof of (i).

(ii), $\Leftarrow$: Suppose X is generic. Then from the proof of (i), we know that $R(x, g)$ does not divide over $\varnothing$ in $T_0 = \mathrm{Th}(M_0)$, where $M_0 = (S, G, R)$. By Lemma 2.4, any Boolean combination of conjugates in T_0 is consistent in T_0 (I do not understand this), hence $\bigcap g_iX \neq \varnothing$.

⇒: Suppose X is non-generic, so then $R(x,1)$ divides over $\varnothing$ in T_0. Let $q^*(y) \in S_{R^*}(M_0^*)$ be an extension of $\mathrm{tp}_{R^*}(1/\varnothing)$ definable over $\mathrm{acl}(\varnothing)$, where M_0^* is a saturated extension of M_0 and $R^*(y,x) := R(x,y)$.

Claim. In M_0, we can find $(b_i : i < \omega)$ s.t. $b_{i+1} \models q^* \restriction \mathrm{acl}(b_0, \dots, b_i)$.

Proof of Claim. Suppose we have found $b_0, \dots, b_{i-1}$ in M_0. Let $q'(y) = q^* \restriction \mathrm{acl}(b_0, \dots, b_{i-1})$; note that $q'(y)$ is consistent. Now, we want to use the saturation of our original universe $\bar{M}$, so we need to translate $q'(y)$ into a set of sentences $q''(y) \subset L_{\bar{M}}$ so that $b \models q'(y)$ in M_0 if and only if $b \models q''(y)$ in $\bar{M}$.[1] But this is no problem since, by construction, M_0 is 'type-interpretable' in $\bar{M}$, e.g. $M_0 \models R(x,y)$ iff $\bar{M} \models x \in y.X$. By definition, $\bar{M}$ is big enough for all our needs, so $q''(y)$ will be a small collection of formulas over a small set of parameters. By saturation of $\bar{M}$, $q''(y)$ is realised in $\bar{M}$, so $q'(y)$ is realised in M_0, and we let $b_i \in M_0$ witness this, proving the claim. □

In fact, this claim and proof hold for any 'small' (but big enough for our purposes) ordinal κ, i.e. we can find $(b_\alpha : \alpha < \kappa)$ such that $b_\alpha \models q^* \restriction \mathrm{acl}(b_\delta : \delta < \alpha)$. Noting that Lemma 2.5 can also be extended to arbitrary κ, we use it to conclude that their exists $k < \omega$ such that $\{R(x, b_\alpha) : \alpha < \kappa\}$ is k-inconsistent in T_0. Now we return to $\bar{M}$. By Erdös–Rado, there is a sequence $c_i : i < \omega$ which is indiscernible over B s.t. $\forall n, \mathrm{tp}(c_1 ... c_n/B) = \mathrm{tp}(b_{\alpha_1} ... b_{\alpha_n}/B)$ for some $\alpha_1 < \ldots < \alpha_n < \kappa$. (Explicitly, you first colour the 1-tuples b_α of the original sequence by $\mathrm{tp}(b_\alpha/B)$. Then by Erdös–Rado and by choosing κ big enough, there exists a monochromatic subset, i.e. we obtain a subsequence of length $\kappa' < \kappa$ so that the type of 1-tuples is constant. We then colour the 2-tuples by their type, and so by Erdös–Rado and by choosing κ' big enough, we obtain a subsequence of length $\kappa'' < \kappa'$ where the type of 1- and 2-tuples is constant. Repeat for all $n < \omega$).

Since X is relatively definable *over* B, the fact that $\bigcap_{i=1}^{k} b_{\alpha_i} X = \varnothing$ is a part of the type over B of $b_{\alpha_1} ... b_{\alpha_n}$. Therefore, $\bigcap_{i=1}^{k} c_i X = \varnothing$, so in particular, $\{c_i X : i < \omega\}$ is inconsistent. □

Now that we know the set of global generic types, Y say, is non-empty, we discuss the action (G, Y). There are two (equivalent) ways to see how G acts naturally on Y. One way is to let $g \in G, p \in S_S(\bar{M})$ and $a \models p$ in some $\bar{M}'$ extending $\bar{M}$ – then $gp := \mathrm{tp}(ga/\bar{M})$. The alternative is to note that since G acts on S, G acts on the Boolean algebra of subsets of S, so G acts on the ultrafilters on this Boolean algebra, and these ultrafilters *are* the global types over S.

Before continuing, we again need to prove a result about local stability:

Lemma 2.7. *Let T be any theory, $\varphi(x,y)$ be stable, $p(x) \in S_\varphi(A), B = \mathrm{acl}^{\mathrm{eq}}(A)$. Let $X = \{q(x) \in S_\varphi(B) : p(x) \subseteq q(x)\}$. Then:*

(i) The group of elementary permutations of B/A acts transitively on X.

(ii) There is an A-definable equivalence relation $E(x_1, x_2)$ with finite many classes each defined by a φ-formula s.t. E distinguishes elements of X, i.e. $q_1(x) = q_2(x)$ iff $q_1(x_1) \cup q_2(x_2) \models E(x_1, x_2)$.

Proof. (i) Let $q(x) \in X, p_1(x) \in S(A)$ extend $p(x) \in S_\varphi(A)$. Then we claim that $q(x) \cup p_1(x)$ is consistent. If the claim failed, then $\psi(x) \cup p_1(x)$ is inconsistent for some $\psi(x) \in q(x)$. By considering A-automorphisms, this means that each of the finitely many A-conjugates of $\psi(x)$ is inconsistent with $p_1(x)$, so their disjunction $\psi'(x)$ is inconsistent with $p_1(x)$. However, $\psi'(x)$ is a φ-formula over A and is in $q(x)$, so it must also be in $p(x)$, contradicting $p_1(x)$ extending $p(x)$.

So, for all $q_1, q_2 \in X$, there exist $a_1 \models q_1, a_2 \models q_2$ such that $\mathrm{tp}(a_1/A) = p_1(x) = \mathrm{tp}(a_2/A)$. So we can find $f \in \mathrm{Aut}(\bar{M}/A)$ such that $f(a_1) = a_2$, so the elementary permutation induced by f takes q_1 to q_2.

(ii) Fix $q(x) \in X$. It has a unique extension to $q'(x) \in S_\varphi(\bar{M})$ definable over B; let $\chi(y)$ be the definition. By (i) and uniqueness (i.e. any $r(x) \in S_\varphi(\bar{M})$ which extends $p(x)$ and is definable over $\mathrm{acl}(A)$ is determined by $r \restriction \mathrm{acl}(A)$), every non-forking global extension of $p(x)$ is definable by some

[1] Keep in mind that we are identifying the domain (S, G) of M_0 with S and G in $\bar{M}$, so using the same element b for both structures does indeed make sense (as long as $b \in S$ or G).

A-conjugate of $\chi(y)$. As $\chi(y)$ has only finitely many conjugates, there are only finitely many such extensions of $p(x)$.

So, let $\Phi(x)$ be a finite collection of ϕ-formulas over B separating the types in X; WMA that Φ is closed under elementary permutations of B/A. Let $E(x_1,x_2) := \bigwedge\{\delta(x_1) \leftrightarrow \delta(x_2) : \delta \in \Phi\}$. □

Proposition 2.8. *Let (G,S) be a ∞-definable/$\varnothing$ homogeneous space in a stable theory T. Let $Y = \{p \in S_S(\bar{M}) : p \text{ generic}\}$.*[2] *Then:*

(i) $G(\bar{M})$ acts transitively on Y.

(ii) For $p(x) \in S_S(\bar{M}), p(x) \in Y$ iff $\forall g \in G, gp$ does not divide over $\varnothing$.

Proof. (i) From Proposition 2.6, $Y \neq \varnothing$. Consider the auxiliary structure defined as follows. For all $\varphi(x,z) \in L$, where x has sort S, let $\Gamma_\varphi \subseteq \mathcal{P}(S)$ be $\{S \cap \varphi(gx,b)(\bar{M}) : g \in G, b \in \bar{M}\}$. Let $\in_\varphi \subseteq S \times \Gamma_\varphi$ be the relation which holds if $x \in S$ is an element of $X \in \Gamma_\varphi$. Let $M_1 = (S, \Gamma_\varphi, \in_\varphi)_{\varphi(x,z)\in L}$, where S and Γ_φ are sorts.[3] Let $T_1 = \mathrm{Th}(M_1)$. Be aware that T_1 is not necessarily stable and M_1 is not necessarily saturated.

There is a natural identification between $S_S(\bar{M})$ and $S_{qf\,S}(M_1)$ (complete quantifier-free types), namely, $p \in S_S(\bar{M})$ is identified with $p' = \{x \in_\varphi S \cap \varphi(gx,b)(\bar{M}) : \varphi(gx,b) \in p(x)\}$. $(*)$

The following remark collects properties of T_1.

Remark 2.9. *(i) Each $\in_\varphi$ is stable in T_1.*

(ii) For all $g \in G$ the mapping $x \mapsto gx$ for $x \in S$ and $X \mapsto gX$ for $X \in \Gamma_\varphi$ is an automorphism of M_1. (So despite the fact G is not a sort in M_1, M_1 does still contain information about G).

(iii) All elements of S have the same type over $\varnothing$ in M_1 (and in T_1), i.e. '$x \in S$' determines a complete type over $\varnothing$ in T_1.

Proof of Remark. (i) Suppose not, so there exists $(s_i \in S, X_i \in \Gamma_\varphi : i < \omega)$ such that $s_i \in_\varphi X_j$ if and only if $i \leq j$. This means that in $\bar{M}$, there exists $(s_i \in S, g_i \in G, b_i \in \bar{M} : i < \omega)$ such that $s_i \in \varphi(g_j x, b_j)(\bar{M})$ iff $i \leq j$, i.e. $\varphi(g_j s_i, b_j)$ iff $i \leq j$, contradicting stability of T.

(ii) Trivial

(iii) By (ii) and the fact that G acts transitively on S (c.f. proof of same result for M_0 in Proposition 2.6). □

Back to proving 2.8(i). Let Y' be the set of $p'(x) \in S_{qf\,S}(M_1)$ where $p'(x)$ does not divide over $\varnothing$, i.e. $p'(x)$ is definable over $\mathrm{acl}(\varnothing)$, i.e. for all $\varphi(x,z), p'(x)|\in_\varphi$ is definable over $\mathrm{acl}(\varnothing)$ in T_1. Now, to show that G acts transitively on Y, it suffices to show that G acts transitively on Y' and that $Y' = Y$ via the identification $(*)$.

Claim. G acts transitively on Y'.

Proof of Claim. From 2.9(iii), we know there is a unique 1-type over $\varnothing$ in S in T_1, $p'_0(x)$ say. By applying Lemma 2.7(ii) to $T_1, \in_\varphi$ (which is stable by 2.9(i)) and $p'_0|\in_\varphi \in S_{\in_\varphi}(\varnothing)$, we conclude that there exists a $\varnothing$-definable equivalence relation $E_\varphi(x_1,x_2)$ on S which distinguishes the types in the set $X_\varphi = \{q(x) \in S_{\in_\varphi}(\mathrm{acl}(\varnothing)) : p'_0|\in_\varphi \subseteq q(x)\}$. Since E_φ is definable in M_1, it is G-invariant by 2.9(ii), so Gacts on its equivalence classes. Since M_1 is 'type-interpretable' in $\bar{M}$, [4] E_φ is relatively definable in $\bar{M}$.By 2.7(i), G acts transitively on X_φ and since E_φ distinguishes elements of X_φ, this implies that G acts transitively on the E_φ classes. (Somehow) G also acts transitively on finite intersections of E_φ, and so by compactness, and working in the saturated model $\bar{M}$ in theory T, G acts transitively on $E = \bigcap_{\varphi \in L} E_\varphi$.

Now, for any $p' \in Y'$ and any φ, we have that $p'|\in_\varphi$ is a type in X_φ: first, $p' \restriction \varnothing = p'_0$ by uniqueness of p'_0 (2.7(iii)) so $p'|\in_\varphi \supseteq p'_0|\in_\varphi$, and second, $p'|\in_\varphi$ is definable over $\mathrm{acl}(\varnothing)$ by definition of Y'. Conversely, picking some $q_\varphi \in X_\varphi$ for every φ will determine a unique $p' \in Y'$, since any formula in p' is quantifier free and so a Boolean combination of φ's. Recalling that an equivalence

[2] Note that '∞-definable/$\varnothing$' is an abbreviation for '∞-definable over $\varnothing$'.

[3] So note, for clarity's sake, that elements of M_1 are either elements of S, or, subsets of S of the form $S \cap \varphi(gx,b)(\bar{M})$ for some $\varphi(x,z), g \in G, b \in \bar{M}$.

[4] M_1 is not necessarily interpretable in $\bar{M}$ because, for example, S can be ∞-definable but not definable in $\bar{M}$.

class of E_φ determines a type in X_φ, we deduce that an equivalence class of E will determine a type in X_φ for every φ, which we have just shown is equivalent to determining a type in Y'. Since G acts transitively on E, we thus conclude that G acts transitively on Y'. □

Claim. $Y = Y'$ via the identification $(*)$.

Proof of Claim. It suffices to show that for any $\varphi(gx, b)$, $X := S \cap \varphi(gx, b)(\bar{M})$ is generic iff "$x \in_\varphi \varphi(gx, b)$" is in some $p' \in Y_1$. For the only-if direction, suppose X is generic so there exists $g_1, \ldots, g_n \in G$ s.t. $g_1X \cup \ldots \cup g_nX = S$. Now fix $p' \in Y_1$. Then for some $1 \leq i \leq n$, "$x \in_\varphi \varphi(gg_i^{-1}x, b)$" is in p', but then "$x \in_\varphi \varphi(gx, b)$" is in $g_i^{-1}p'$.

Conversely, let "$x \in_\varphi \varphi(gx, b)'' \in p' \in Y$. By 2.7(ii), $\{p''\restriction\in_\varphi : p'' \in Y'\}$ is finite and G acts transitively on this set – let $g_1, \ldots, g_n$ witness this transitivity. Hence, "$x \in_\varphi \bigvee_{1\leq i\leq n} \varphi(gg_ix, b)$" is in every element of Y'. Letting $X' := S \cap (\bigvee_{1\leq i\leq n} \varphi(gg_ix, b))(\bar{M})$ and using the only-if direction, we see that the complement of X' is non-generic, so by Proposition 2.6(i), X' is generic. But X' is a finite union of G-translates of X (note that $g_i^{-1}X = S \cap \varphi(gg_ix, b)(\bar{M})$), so X is generic too. This ends the claim and so the proof of (i). □

(ii) $\Leftarrow$: If $p(x) \in Y$, then $gp \in Y$ for all G, so it suffices to show $p(x)$ does not divide over $\varnothing$ in T_1. Suppose otherwise, so that there exists $\varphi(x, b) \in p(x)$ and $(b_i : i < \kappa)$ indiscernible over $\varnothing$ s.t. $\{\varphi(x, b_i) : i < \kappa\}$ is k-inconsistent for some finite k. By indiscernibility, there exists an automorphism f_i sending b to b_i for each $i < \kappa$. Then, by k-inconsistency, there are κ many distinct types amongst $f_i(p)$, i.e. p has κ many conjugates over $\mathrm{Aut}(\bar{M})$. Using the correspondence $(*)$ from proof of (i), we then get that p' has κ many conjugates under $\mathrm{Aut}(M_1)$. By choosing κ large enough, this contradicts definability of p' over $\mathrm{acl}(\varnothing)$ in M_1.

$\Rightarrow$: Suppose $p(x) \in S_S(\bar{M})$ is not generic, so there exists formula in $\psi(x) \in p(x)$ such that $X = S \cap \psi(\bar{M})$ is not generic. By 2.6(ii) (and standard compactness/saturation argument), there exists an indiscernible (over parameters defining X) $(g_i : i < \kappa) \subset G$ s.t. $\{g_iX : i < \kappa\}$ is k-inconsistent for some finite k. So among $\{g_ip : i < \kappa\}$ there are κ distinct types, so they cannot all be definable over $\mathrm{acl}(\varnothing)$ (by choosing κ large enough), i.e. gp is not definable over $\mathrm{acl}(\varnothing)$ for some $g \in G$, so (since T is stable), gp divides over $\varnothing$. □

Proposition 2.10. *Given a stable theory T, let (G, S) a type definable pair over the empty set an homogeneous space. Let $G_\emptyset$ de the intersection of all relatively definable over $\emptyset$ subgroups of G of finite index. Then for every generic $p(x) \in S_S(\bar{M})$ and $g \in G_\emptyset^0$, $g \cdot p = p$.*

Proof. Fix a formula $\varphi(x, z)$. Let E_φ be the finite equivalence relation given by 2.7(ii) (for E_φ,$p_0\restriction\Sigma_\varphi$). In E_φ each class is defined by 2.7(ii). Note that E_φ is G-invariant.(check).

Let G_φ^0 be the fixator of S/E_φ, i.e., the subgroup of G that fixes all E_φ classes. E_φ^0 is a relatively $\emptyset$-definable normal subgroup of G of finite index in $\bar{\mathcal{M}}$ (check). By 2.7, G_φ^0 fixes every global extension of $p_0\restriction E_\varphi$ to $S_{q.f.S}(\bar{\mathcal{M}}_1)$, where p_0 is the unique type over $\emptyset$ in T_1. So $\bigcap_\varphi G_\varphi^0$ fixes every element of Y_1. This implies G_φ^0 fixes every $p \in Y$ (via some translation). And since $G_\phi \subseteq \bigcap_\varphi G_\varphi^0$ fixes every $p \in Y$. □

Corollary 2.11. *Given T be a stable theory. Let G be any ∞-definable group definable over $\emptyset$. Then every relatively definable (with parameters) subgroup of finite index contains a relatively definable over $\emptyset$ subgroup of finite index.*

Proof. We can apply 2.10 to the action of G on itself by left multiplication. Let $p(x) \in S_G(\bar{\mathcal{M}})$ and H be a relatively definable subgroup of G of finite. Then there are finitely many left cosets of H in G. Hence for every $x \in G$ we have $x \in a_1H \vee a_2H \vee \ldots \vee a_nH$.

So $p(x)$ determines one of these cosets, say a_iH, i.e. $\forall a \models p(x)\ a \in a_iH$.(check) If $gp = p$ then $g \in H$. But since p is generic $gp = p$ for all $g \in G_\emptyset^0$, so $G_\emptyset^0 \subseteq H$. By compactness there exists some finite intersection H_0 of relatively definable over $\emptyset$ subgroups of finite index so $H_0 \subseteq H$. □

Let us note that given G a ∞-definable group over $\emptyset$ and a small set A of parameters we can also define G_A^0 as the intersection of all the relatively definable over A subgroups of finite index. and the last result says that G_A^0 doesn't depend on A, i.e., $G_A^0 = G_\emptyset^0$. In general if $A \subseteq B$ then

$G_B^0 \subseteq G_A^0 \subseteq G_\emptyset^0$. So in the stable case we just call $G_\emptyset^0 = G^0$ the connected component of G which is the intersection of all relatively definable subgroups of finite index. G^0 is normal, in fact every relatively definable subgroup of finite index contains a relatively definable normal subgroup of finite index, so $G_0 = \bigcap_{i \in I} H_i$ with H_i normal of finite index relative definable over $\emptyset$.

Hence $G/G^0 = \underleftarrow{lim} G/H_i$ is a profinite group.

Consider for example $T = Th(\mathbb{Z}, +)$, $M_0 = (\mathbb{Z}, +)$ and $\bar{\mathcal{M}}$ a saturated model. Then $G^0 = \bigcap_n nG$ and $G^0(M_0) = 0$ but $G/G_0 = \widehat{\mathbb{Z}} = \underleftarrow{lim} \mathbb{Z}/n\mathbb{Z}$.

In general if G is a ∞-definable over $\emptyset$ group we can define G_A^{00} for any small set A of parameters as the smallest ∞-definable over A subgroup of G of "bounded" index, i.e. less than $2^{|T|+A}$, this means that in general $G_A^{00} < G_A^0$ by the proof of the corollary above, one can also show that any gobal type determines a coset of G_A^{00}, so the same proof shows that in the stable case $G_A^{00} = G_\emptyset^{00} = G^0$.

Stab(p)/ $Stab_{\varphi^{-1}}$

Definition 2.12. Let (G, S) be ∞-definable over $\emptyset$. And T stable theory.

(i) Let $p \in S_S(\bar{\mathcal{M}})$. By stabilizer of p, in symbols $Stab(p)$, we mean $\{g \in G : gp = p\}$.

(ii) If $p \in S_S(A)$ with A stationary, i.e. has no forking extension, $stab(p)$ is by definition $Stab(p \restriction \bar{\mathcal{M}})$ where $p \restriction \bar{\mathcal{M}}$ is the unique global extension of p definable in A.

(iii) Fix $p(x) \in S_S(\bar{\mathcal{M}})$ and $\varphi(x, z)$ an $\mathcal{L}$- formula. Let $\varphi'(x, y, z)$, with variable x of sort S and variable y of sort G, be $\varphi(x \cdot y, z)$. Let $\delta(y, z)$ the φ-definition of $p(x)$. Define $Stab_{\varphi'}(p) = \{g \in G : \forall y, z \ \delta(y \cdot g, z) \leftrightarrow \delta(y, z)\}$ a relatively definable subgroup of G.

Proposition 2.13. *Let $p(x) \in S_S(\bar{\mathcal{M}}_1)$.*

1. *$Stab(p) = \bigcap_\varphi Stab_{\varphi'}(p)$.*

2. *p is generic if and only if $G^0 \subseteq Stab(p)$.*

Proof. Part 1 follows from the definitions.

Consider for part 2 $g \in G^0$. If p is generic then $G^0 \subseteq Stab(p)$ by 2.10. On the other hand, if p is non generic then we know that for any k there is an indiscernible sequence $\{g_i\}$ $i < k$ in G such that $g_i p \neq g_j p$ for all $i \neq j$ by 2.6. And $gp = hp$ if and only if $h^{-1}gp = p$, hence all the g_i are in different cosets with respect to $Stab(p)$. So $Stab(p)$ has unbounded index in G, so it can't contain G^0 □

Remark 2.14. *Assume $G = S$. for every p generic we have that $Stab(p) = G^0$. In fact there is a bijection h between G/G^0 and Y, the set of generic types, i.e. each coset of G^0 in G contains a unique generic type, $Y \subseteq S_G(\bar{\mathcal{M}})$ is closed and h determines a homeomorphism between G/G^0 and Y.*

Note that Y the space of global generic types is closed under G, i.e. $gp \in Y$ if $p \in Y$. G acts transitively on Y since elements of G act by homeomorphism of Y and we can represent G/G^0 as the group of automorphism of (G,Y), i.e. the autohomeomorphisms of Y which commute with the action of G.

Remark 2.15. *Given T a stable theory. (G, S). Let $p(x) \in S_S(A)$ stationary. $Stab(p) = \{g \in G : (a \models p \wedge a \mathop{\smile\!\!\!\!\!\mid}_A g) \rightarrow g \cdot a \models p\}$.*

Chapter 3

Generalizations

We continue now outside stable theories on other environments where the notions we have been discussing can make sense, probably with some adjustments, for example simple theories, NIP theories, pseudofinite theories or NTP2 theories.

We will start considering Simple theories. Here we take as references the papers by B. Kim and A. Pillay [4] and by A. Pillay [6], also the book from F. Wagner [7].

We consider a complete theory T, $\bar{\mathcal{M}}$ a saturated enough "big" model, $A, B \subset \bar{\mathcal{M}}$ "small" with respect to $\bar{\mathcal{M}}$. M, $N < \bar{\mathcal{M}}$ models of smaller cardinality.

Definition 3.1. A theory T is simple if every type $p(x) \in S(B)$ does not divide over some $A \subseteq B$ with $|A| \leq |T|$. *Simple theory*

Equivalently, T does not have the tree property. Where T has the tree property if there is a formula $\varphi(x, y)$, $k < \omega$ and b_η for $\eta \in \omega^{<\omega}$, such that for all $\eta \in \omega^{<\omega}$ the set $\{\varphi(x, b_{\eta^\frown(i)}) : i < \omega\}$ is k-inconsistent but if $\tau \in \omega^\omega$ the set $\{\varphi(x, b_{\tau\restriction n}) : n \in \omega\}$ is consistent.

For a simple theory T the algebraic properties of dividing in stable theories still hold but not the so called "multiplicity theory".

Proposition 3.2. *Suppose T is simple. Define $a \underset{C}{\downarrow} b$ if $tp(a/bC)$ does not divide over C. Then*

Existence *For all a, and $A \subseteq B$ there exists a' such that $a \underset{A}{\equiv} a'$ and $a' \underset{A}{\downarrow} B$.*

Symmetry *$a \underset{A}{\downarrow} b$ if and only if $b \underset{A}{\downarrow} a$.*

Transitivity *Given $A \subseteq B \subseteq C$. $a \underset{A}{\downarrow} B$ and $a \underset{B}{\downarrow} C$ if and only if $a \underset{A}{\downarrow} C$*

- *Suppose $a \underset{A}{\downarrow} b$ and $(b_i : i < \omega)$ is independent over A, and $b = b_0$. Then there exists a' with $a \underset{A}{\downarrow} A \cup \{b_i : i < \omega\}$ and $a'b_i \underset{A}{\equiv} ab_i$ for every i.*

Forking equals dividing *$\varphi(x, b)$ divides over A if and only if $\varphi(x, b)$ forks over A.*

Recall that if T is stable, then every type over a model is stationary, i.e. has a unique non forking global extension. In particular if $M \subseteq A$, $M \subseteq B$ and $p_1(x) \in S(A)$ and $p_2(x) \in S(B)$ are non forking extensions of p then $p_1(x) \cup p_2(x)$ extends to a non forking extension of p over $A \cup B$. This has an extension to simple theories.

Proposition 3.3 (Independence Theorem)**.** *Consider T simple. Let $p(x) \in S(M)$ and let $M \subseteq A$, $M \subseteq B$ be such that $A \underset{M}{\downarrow} B$. Suppose $p_1(x) \in S(A)$ and $p_2(x) \in S(B)$ are non forking extensions of p then there exists $q(x) \in S(A \cup B)$ a non forking extension of p such that $q\restriction A = p_1$ and $q\restriction B = p_2$*

Basic examples of unstables theories include the random graph, pseudo-finite fields, ACFA and other suitable stable theories with a random relation. (T a stable theory with quantifier elimination plus a mild condition and a new n-place relation R, T_R has a model companion, and each completion is simple).

If we consider the random graph with $R(x, y)$, $\bar{\mathcal{M}}$ a big model. Any non algebraic 1-type over $\bar{\mathcal{M}}$ does not divide over $\emptyset$. And p_0 has as many as you want non algebraic extensions over $\bar{\mathcal{M}}$ all of them non forking.

Consider now a $\emptyset$-definable group G. (More generally a ∞-definable homogeneous space (G, S) over $\emptyset$)

f-generic types **Definition 3.4.** Let $p(x) \in S_G(A)$, $p(x) \vdash$ "$x \in G$". Call $p(x)$ left f-generic if for all $g \in G$ and $a \models p$ such that $a \underset{A}{\downarrow} g$ then $g \cdot a \underset{\emptyset}{\downarrow} A, g$

Theorem 3.5. *Given a simple theory T, and G a $\emptyset$-definable group.*

1. *Let $p(x) \in S_G(A)$, $q(x) \in S_G(B)$, $A \subseteq B$, $p \subseteq q$. Then p is left f-generic if and only if q is left f-generic.*

2. *Left f-generic types exists. (This also holds in ∞-definable (G, S) context)*

3. *Left f-generic equals right f-generic.*

4. *If $p(x) \in S(A)$ is f-generic if and only if for all $g \in G$, $a \models p$, $a \underset{A}{\downarrow} g$ then $g \cdot a \underset{A}{\downarrow} g$.*

Methods for part 2 involve an invariant version of D-rank

We call a partial type $\Phi(x)$ over A ($\Phi(x) \wedge x \in G$) left f-generic if Φ extends to a complete left f-generic type over A.

Lemma 3.6. *Let T be a simple theory, G a group and $\Phi(x)$ a partial type over A implying $x \in G$. The following are equivalent.*

1. *$\Phi(x)$ is f-generic.*

2. *$\forall g \in G$ $g\Phi(x)$ does not fork over $\emptyset$. (Where $g\Phi(x)$is $\Phi(g^{-1}x)$).*

3. *$\forall g \in G$ $g\Phi(x)$ does not fork over A.*

This shows that in the stable case f-generic is equivalent to generic, say for a global type by 2.8.

Proof. Assume (1). Let $p(x) \supseteq \Phi(x)$, $p(x) \in S(A)$ be f-generic. Let $g \in G$ and $a \models p$ with $a \downarrow g$ so $g \cdot a \underset{\emptyset}{\downarrow} (A, g)$ but $g \cdot a$ satisfies $g \cdot \Phi(x)(= \Phi(g^{-1}x))$, which is a partial type over (A, g). So $g \cdot \Phi(x)$ does not fork over $\emptyset$ as it has an extension to a complete type $\text{tp}(g \cdot a/A, g)$ which does not fork over A. Hence (2) follows.

(3) follows form (2) from the definitions.

Assume now (3) to prove (1). Let g realise some $q(x) \in S_G(A)$ which is f-generic. By (3) $g\Phi$ does not fork over A. (if $g\Phi$ does not fork then it can be extended to a compete type that does not fork over $\emptyset$) So exists d such that $d \models \Phi$ and $\text{tp}(g \cdot d/A, g)$ does not fork over A. So by 3.5 since $g \underset{A}{\downarrow} g \cdot d$, $\text{tp}(g/A, g \cdot d)$ is f-generic, this give us that $tp(g/A, (g \cdot d)^{-1})$ is f-generic and hence $tp((g \cdot d)^{-1}g/A, (g \cdot d)^{-1})$ is f-generic. (See Remark 3.2 in [6]). This means that $\text{tp}(d^{-1}/A, (g \cdot d)^{-1})$ is generic and because of translation does not fork over $\emptyset$ this does not fork over A, so $\text{tp}(d^{-1}/A)$ is f-generic by 3.5. So $\text{tp}(d/A)$ is f-generic, since inverting d changes left generics by right generics. But $\Phi(d)$ and Φ is over A so Φ if f-generic. □

We proceed now to a brief discussion about stabilizers. Considering henceforth T simple and G being $\emptyset$-definable.

Definition 3.7. Let $p(x) \in S_G(A)$. $S(p) = \{g \in G : gp(x) \cup p(x)\ does\ not\ fork\ over\ A\}$. In other words, all $g \in G$ such that $\exists c \models p$, $c \underset{A}{\downarrow} g$ and $c = g \cdot b$ for some $b \models p$.

Remark In fact using invariant D,φ rank arguments as in Remark 4.2 in "Definability and definable groups" $b \underset{A}{\downarrow} g$ so in fact $S(p) = \{g \in G : \exists b, c \models p, b \downarrow g, c \downarrow g, c = G \cdot b\}$.

Proposition 3.8. *(i) $S(p)$ is type definable over A. and if A is a model $A = \mathcal{M} < \bar{\mathcal{M}}$ then $S(p).S(p) = Stab(p)$ is a type definable over A subgroup G.*

(ii) $p(x) \in S_G(\mathcal{M})$ is f-generic if and only if $Stab(p) = G^{00}_M$ has bounded index in G (i.e. index is less than $2^{|T|+|M|} < \kappa$).

3.1 Amenability, Measures, NIP

During this section the main references are A. Pillay and E. Hrushovski paper about NIP theories, [3] and also [2].

Definition 3.9. *Keisler measures*

(i) Let T be any theory, $\bar{\mathcal{M}}$ a "big" model, $\mathcal{M} < \bar{\mathcal{M}}$ small. X an $\mathcal{M}$-definable set in $\bar{\mathcal{M}}$. A Keisler measure on X over M is a finitely additive probability measure on $Def_X(\mathcal{M})$, i.e. definable over $\mathcal{M}$ subsets of X, equivalently definable subsets of $X(M)$, i.e. $\mu(X) = 1$, $\mu(\emptyset) = 0$, $\mu(Y) \in [0, 1]$ for all $\mathcal{M}$-definable $Y \subseteq X$. If Y_1, Y_2 are disjoint then $\mu(Y_1 \cup Y_2) = \mu(Y_1) + \mu(Y_2)$. **Note:** When $\mathcal{M} = \bar{\mathcal{M}}$ we call μ a global Keisler measure on X. Also a special case of the Keisler measure on X over $\mathcal{M}$ is a complete type $p(x) \in S_X(\mathcal{M})$, its values are in $\{0, 1\}$.

(ii) Suppose X is a group G over $\mathcal{M}$ and μ is a Keisler measure on G over M. Call μ (left) $G(\mathcal{M})$-invariant if for all definable subsets $Y \subseteq G(\mathcal{M})$ (or $Y \subseteq G$ definable over $\mathcal{M}$) and $g \in G(\mathcal{M})$ we have $\mu(gY) = \mu(Y)$.

Recall that an abstract group (a discrete group), H, is amenable if there is a finitely additive left invariant probability measure on $\mathcal{P}(H)$.

Definition 3.10. Let T be any theory, $\bar{\mathcal{M}}$ a "big" model, $\mathcal{M} < \bar{\mathcal{M}}$ small. G an $\mathcal{M}$-definable group in $\bar{\mathcal{M}}$. We call $G(\mathcal{M})$ definably amenable if there exists a left $G(\mathcal{M})$-invariant Keisler measure in G over $\mathcal{M}$. *Definably amenable groups*

Fact. It follows from 5.6 in [3] that $G(M)$ is definable amenable if and only if $G(\bar{\mathcal{M}})$ is definably amenable, if $G(N)$ definably amenable for all $\mathcal{N}$ in which G is defined. Note also that if $G(M)$ is amenable as a discrete group then it is definably amenable. And if T is stable then every group G is definably amenable. Consider the canonical surjection between G and the profinite compact topological group G/G^0 where, as usual, G^0 is the intersection of all $\emptyset$-definable subgroups of finite index. G/G^0 has a unique Haar measure $\mathcal{H}$[1], an invariant Borel probability measure. Let $X \subseteq G$ be definable, define $\mu(X) = \mathcal{H}(\pi(X)) \subseteq G/G^0$.
Question For T simple is any definable group G definably amenable?

Example 3.11. Let $T = RCF$, $\bar{\mathcal{M}} = (K, +, \cdot)$, $\mathcal{M} = (\mathbb{R}, +, \cdot) < \bar{\mathcal{M}}$.

(a) Let $G \subseteq K^n$ be a $\mathcal{M}$-definable group such that $G(\mathcal{M}) \subseteq \mathbb{R}^n$ with the Euclidean topology is compact (Hausdorff). (we call such groups $G(\mathcal{M})$ compact semi algebraic Lie groups). Then G is definably amenable, in fact there is a unique (global) left G-invariant Keisler measure on $G = G(\bar{\mathcal{M}})$ and it is the unique lifting of the Haar measure on the compact group $G(\mathcal{M})$ to a Keisler measure on G. (e.g. $SO_2(\mathbb{R})$ or $SO_3(\mathbb{R})$)

(b) $SL_2(\mathbb{R})$ is not definably amenable.

[1] Consider a compact topological group Γ and Σ a σ-algebra containing the borel subsets of Γ, i.e. countable unions and intersections of open subsets and complements of these. A Haar measure, $\mathcal{H}$, is a measure from Σ to $[0, \infty)$ such that, $\mathcal{H}(\Gamma) = 1$, and $\mathcal{H}(\gamma S) = \mathcal{H}(S)$ for each $S \in \Sigma$, in other words, is invariant under translations. Moreover, it can be proved that for a compact topological group there exists always a Haar measure.

NIP theory
NTP2 theory

Definition 3.12.

- Let T be a complete theory, T has NIP if for any $\varphi(x,y) \in \mathcal{L}$ and indiscernible sequence $(a_i : i < \omega)$ and $b \in \bar{\mathcal{M}}$, the truth value of $\varphi(a_i, b)$ stabilizes as $i \to \infty$, i.e. there exists n such that $\models \varphi(a_i, b)$ for all $i > n$ or $\models \neg\varphi(a_i, b)$ for all $i > n$. (**Note:** Stable implies NIP)

- T has NTP2 if is not the case that there are a formula $\varphi(x,y)$ and $k < \omega$ and $(b_{i,j} : i, j < \omega)$ such that

  (i) $\forall i \{\varphi(x, b_{i,j}) : j < \omega\}$ is k-inconsistent, i.e. every set of size k in a fix row(index i) is inconsistent.

  (ii) $\forall \eta \in \omega^\omega$ $\{\varphi(x, b_{i,\eta(i)}) : i < \omega\}$ is consistent. **Fact**: NTP2 is a common generalization of simple and NIP.

Proposition 3.13. *Consider T a NIP theory. Let $p(x) \in S(\bar{\mathcal{M}})$, $\mathcal{M} < \bar{\mathcal{M}}$ a small model. The following are equivalent.*

1. *$p(x)$ does not divide over $\mathcal{M}$.*

2. *$p(x)$ does not fork over $\mathcal{M}$.*

3. *$p(x)$ is* $\mathrm{Aut}(\bar{\mathcal{M}}/\mathcal{M})$*-invariant. (i.e. given $\varphi(x,y)$ whether or not $\varphi(x,y) \in p$ depends on* $\mathrm{tp}(b/\mathcal{M})$*).*

Proof. To prove the equivalence between parts (1) and (2) we do not use the NIP hypothesis. First, we mention that diving implies forking. Let us now assume that $p(x)$ forks over $\mathcal{M}$. Since $p(x)$ forks there is a formula $\psi(x) \in p(x)$ that forks, i.e. $\psi(x,b)$ implies $\varphi_1(x,b_1) \vee \ldots \vee \varphi_n(x,b_n)$ and each of the formulas $\varphi_i(x,b_i)$ divides over $\mathcal{M}$. Since $p(x)$ is a complete type at least one of the formulas $\varphi_i(x,b_i)$ is in $p(x)$ and $\varphi_i(x,b_i)$ divides over M. Hence $p(x)$ divides over M.

Part (3) always implies part (1) since the type $p(x)$ does not divide when for every $\varphi(x,b) \in p$, and $b = b_1, b_2, \ldots$ indiscernibles over $\mathcal{M}$ then $\{\varphi(x,b_i) : i < \omega\}$ is consistent. But $\mathrm{tp}(b_i/\mathcal{M}) = tp(b_0/\mathcal{M})$ for every i so by part (3) $\varphi(x,b) \in p(x)$ for all i, hence $\{\varphi(x,b_i) : i < \omega\}$ is consistent, and because of this $p(x)$ does not divide.

Assume now part (1) and let $\varphi(x,y) \in L$ and $b, b' \in \bar{\mathcal{M}}$ have the same type over $\mathcal{M}$. We want to show $\varphi(x,b) \in p$ if and only if $\varphi(x,b') \in p$. Now let $q(y) \in S(\bar{\mathcal{M}})$ be a coheir of $\mathrm{tp}(b/\mathcal{M})$, i.e. it extends $\mathrm{tp}(b/\mathcal{M})$ and forks in $\mathcal{M}$. Let $b_1, b_2, \ldots$ such that $b_1 \models q\restriction_{(M,b,b')}$, and $b_{n+1} \models q\restriction_{M,b,b',b_1,\ldots,b_n}$. Then we can build $b, b_1, b_2, \ldots$ and $b', b_1, \ldots$ indiscernibles sequences over $\mathcal{M}$.
Claim $\varphi(x,b) \in p$ if and only if $\varphi(x,b_1) \in p$.

Proof of claim. Suppose not, without loss of generality $\varphi(x,b) \wedge \neg\varphi(x,b_1) \in p(x)$ But then (b, b_1), (b_2, b_3), $\ldots$ is also an indiscernible sequence over $\mathcal{M}$, and since $p(x)$ does not divide over $\mathcal{M}$ and $\varphi(x,b) \wedge \neg\varphi(x,b_1) \in p(x)$ so $\varphi(x,b) \wedge \neg\varphi(x,b_1) \wedge \varphi(x,b_2) \wedge \neg\varphi(x,b_3) \wedge \varphi(x,b_4) \wedge \neg\varphi(x,b_5) \in p(x)$ is consistent. If it is realized by c, using the indiscernibility of (b_i), there is no eventual truth value of the formula $\varphi(c, b_1)$, contradicting NIP. □

So the claim is proved. for the same reason $\varphi(x,b') \in p(x)$ if and only if $\varphi(x,b_1) \in p$. So $\varphi(x,b) \in p$ if and only if $\varphi(x,b') \in p$. □

Remark: There are two extreme cases of an $\mathrm{Aut}(\bar{\mathcal{M}}/\mathcal{M})$-invariant type $p(x) \in S(\bar{\mathcal{M}})$.

(a) $p(x)$ is finitely satisfiable in $\mathcal{M}$.(Every $\varphi(x) \in p$ is realized in $\mathcal{M}$)

(b) $p(x)$ is definable over M.

f-generic types
NIP context

Definition 3.14. Consider a NIP Theory. Let G be a definable group defined over $\mathcal{M}$. Let $p(x) \in S_G(\bar{\mathcal{M}})$. Call $p(x)$ left f-generic with respect to $\mathcal{M}$ if $\forall g \in G = G(\bar{\mathcal{M}})$ the translate $g \cdot p$ does not divide over $\mathcal{M}$. (Equivalently gp is $\mathrm{Aut}(\bar{\mathcal{M}}/\mathcal{M})$-invariant).

Note that in the stable case it agrees with the notion presented in the last chapter but not in the simple case.
Fact: Assuming NIP suppose G is defined over M_1, M_2 then there exists a global f-generic with respect to M_1 if and only if there exists a global f-generic with respect to M_2. (See [2] for more on this)

We now aim towards the next.

Proposition 3.15. *Working within a NIP theory. Let G be definable group then G is definably amenable if and only if G has a global left f-generic type (with respect of some model $\mathcal{M}$).*

Sketch of a proof for one implication. (See 5.8, 5.9 in [2]) Suppose there exists global left invariant Keisler measure. Use the NIP condition (argument of Keisler) to find other such μ which is definable (over some $\mathcal{M}_0$), i.e. for every $\varphi(x,y) \in \mathcal{L}$ and a closed $C \subseteq [0,1]$ the set $\{b : \mu(\varphi(x,b)) \in C\}$ is type definable over $\mathcal{M}_0$. (So μ is $\mathrm{Aut}(\bar{\mathcal{M}}/\mathcal{M}_0)$-invariant, even more for each definable $X \subseteq G$ and $h \in \mathrm{Aut}(\bar{\mathcal{M}}/\mathcal{M})$, $\mu(X \triangle h(X)) = 0$.)

Let $p(x) \in S_G(\bar{\mathcal{M}})$ be in $Supp(\mu)$, i.e. for every $\psi(x) \in p(x)$, $\mu(\psi) > 0$. Likewise as μ is G-invariant, $\forall g \in G$ gp is too an element of $Supp(\mu)$. So $\forall g \in G$ gp is $\mathrm{Aut}(\bar{\mathcal{M}}/\mathcal{M}_0)$ invariant. (Because if $\psi(x) \in gp$ and $h \in \mathrm{Aut}(\bar{\mathcal{M}}/\mathcal{M}_0)$ then $\mu(\psi(x) \triangle h\psi(x)) = 0$) hence $h\psi(x) \in p$. So p is f-generic with respect to $\mathcal{M}_0$.) □

Question Find a formulation of this last proposition for the NTP2 case.

For example $SL_2(\mathbb{R})$ is not definable amenable because it acts on $\mathbb{P}^1(\mathbb{R})$. If there is a invariant measure in $SL_2(\mathbb{R})$ then there is an invariant measure in $\mathbb{P}^1(\mathbb{R})$. (See [3] remark 5.2 (iv))

Remark 3.16. *Two extreme cases*

(a) *There is $p(x) \in S_G(\bar{\mathcal{M}})$, $\mathcal{M} \in \bar{\mathcal{M}}$ such that every $g \cdot p$ is finitely satisfiable in $\mathcal{M}$.(fsg group)*

In this case there is a unique left invariant Keisler measure on G which is also the unique right invariant Keisler measure on G.

Example: *Definable compact group in o-minimal structures. $SO_3(\mathbb{R})$ with $R \models RCF$. (with analogues in p-adics, AVCF)*

(b) *There is $p(x) \in S_G(\mathcal{M})$ f-generic with respect of $\mathcal{M}$ and definable. (Equivalently every gp is definable over $\mathcal{M}$)*

Example: *$(\mathbb{Z}, +, <) < G$. There are two classes of definable f-generics at $\pm\infty$. Any non algebraic type of this will be a definable f-generic. (Additional data are the cosets of nG for all n). This lifts to the multiplicative group of $\mathbb{Q}_p$.*

Consider now any theory T. Some $\bar{\mathcal{M}}$ model of size κ, saturated, with κ strongly inaccesible, G definable group over some small A (size less than κ). G_A^{00} is the smallest type definable over A subgroup of G of index less than κ. (equivalently index less than $2^{|T|+|A|}$).

Fact 3.17.
1. $G_A^{00} \triangleleft G$ and G/G_A^{00} with the logic topology is a compact Hausdorff topological group. The logic topology is the one where $F \subseteq G/G_A^{00}$ is closed if $\pi^{-1}(y) \subseteq F$ is type definable over some small B, where π is the canonical surjection from G to G_A^{00}.
2. G/G_A^{00} has a maximal profinite quotient, precisely G/G_A^0. ($G_A^{00} < G_A^0 < G$)

Example 3.18. Let $T = RCF$. $G = SO_2 = \mathbb{S}$, $M = A = \mathbb{R}$. G_A^{00} is the infinitesimals and G/G_A^{00} identifies with $SO_2(\mathbb{R})$.

Fact 3.19. Consider T a NIP theory. $\mathcal{M}$ countable and $p(x) \in S(\bar{\mathcal{M}})$. p does not fork over $\mathcal{M}$. Then for every $\varphi(x,y) \in \mathcal{L}$ the set $\{b : \varphi(x,b) \in p\}$ is a countable union of sets type definable over $\mathcal{M}$. (See 2.6 of [2])

Also if G is a definable group G_A^{00} does not depend on the choice of A.(See [3])

Proposition 3.20. *In a NIP theory if G is a definable group over $\mathcal{M}_0$ and $p(x) \in S_G(\bar{\mathcal{M}})$ a left f-generic with respect to $\mathcal{M}_0$. Then $Stab(p) := \{g \in G : g \cdot p = p\} = G^{00}$.*

Proof. **Claim 1** If $g_1, g_2 \in G(\bar{\mathcal{M}}) = G$ are such that $\mathrm{tp}(g_1/\mathcal{M}_0) = \mathrm{tp}(g_2/\mathcal{M}_0)$ then $g_2^{-1}g_1 \in Stab(p)$.

Proof of Claim 1. Both p and gp are $\mathrm{Aut}(\bar{\mathcal{M}}/\mathcal{M}_0)$-invariant. Let $f \in \mathrm{Aut}(\bar{\mathcal{M}}/\mathcal{M})$ such that $f(g_1) = g_2$ so $g_1 p = f(g_1 p) = f(g_1)f(p) = g_2 f(p) = g_2 p$. □

Claim 2 Suppose $h \in Stab(p)$ then $h = g_1^{-1} g_2$ for some g_1, g_2 with the same type over $\mathcal{M}_0$.

Proof of Claim 2. Since $hp = p$ then $ha = b$ where $a, b \models p \in \bar{\bar{\mathcal{M}}}$. Let $\Sigma(x, y, h) := \exists x, y \; x \underset{\mathcal{M}_0}{\equiv} y \wedge h = xy^{-1}$, as $\bar{\mathcal{M}}$ is saturated there exists $g_1, g_2 \in \mathcal{M}$ with $\Sigma(g_1, g_2, h)$ so $h = g_1 g_2^{-1}$. □

So by claim 2 $Stabl(p)$ is type definable over $\mathcal{M}_0$ and $\{xy^{-1} : x \underset{\mathcal{M}_0}{\equiv} y, x, y \in G\}$. By claim 1 if $\mathrm{tp}(g_1/\mathcal{M}_0) = \mathrm{tp}(g_2/\mathcal{M}_0)$ then $g_1 Stabl(p) = g_2 Stabl(p)$ so index of $Stabl(p)$ in G is bounded ($Stabl(p)$ is bounded index type definable over $\mathcal{M}_0$).

But for any type $p(x)$, $p(x)$ determines a coset of G^{00}, i.e. $a, b \models p$. Then $ab^{-1} \in G^{00}$. If $a \models p$ then $aG^{00}(\bar{\bar{\mathcal{M}}}) \cap G(\bar{\mathcal{M}}) \neq \emptyset$ so if $a, b \models p$ then $ab^{-1} \in G^{00}$, therefore $Stabl(p) < G^{00}$, hence $Stab(p) = G^{00}$. □

Proposition 3.21. *Let T be a NIP theory. If G has a global left f-generic p then G is definably amenable.*

Proof. We try to define our left G-invariant Keisler measure on $Def_G(\bar{\mathcal{M}})$ (the definable subsets of G). We can reduce to the case of T countable and p left generic with respect to f and a countable model $\mathcal{M}_0$. Let $X \subseteq G(\bar{\mathcal{M}})$ be definable. To define $\mu(X)$ we may assume X is defined over $\mathcal{M}_0$. By 3.19 $Y = \{g \in G : X \in gp\}$ by countability this is a countable union of type definable over $\mathcal{M}_0$ sets. By 3.20 whether or not $g \in Y$ depends only on $gG^{00} < G$, i.e. $g \in Y$ and $h \in gG^{00}$ implies that $h \in Y$, so we consider $\pi : G \longrightarrow G/G^{00}$, and consider $Z = \pi(Y)$. Y is the preimage of Z so $\pi^{-1}(Z)$ is a countable union of closed sets in logic topology, so Borel subset of subset of G/G^{00}. G/G^{00} as a compact group has a unique left invariant Borel probability measure $\mathcal{H}$ (Haar measure).

Define $\mu(X)$ to be $\mathcal{H}(Z) = \mathcal{H}(\pi(Y))$. □

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